Introduction to Algebraic Topology through CW-Complexes

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Aim of the course

Algebraic Topology stands as one of the most important branches of contemporary mathematics, with applications popping up in other fields like Differential Geometry, Dynamical Systems, Analysis, and Algebra. At its core, it's about tackling topological problems using tools from algebra.

While a typical course on this subject often delves into highly intricate and abstract arguments, we'll take a slightly different route. We'll focus on a special class of topological spaces known as *CW-Complexes*. For these spaces, many key results can be reached through more combinatorial arguments, which are often more intuitive and geometric. The great thing is that many important topological spaces are actually CW-complexes! For instance, the torus, wedge sums of spheres, projective spaces, compact Lie groups, and even some linear groups like $\mathrm{SL}_n(\mathbb{R})$ fall into this category. One of the main points of the study of CW-complexes is the construction of the so called *Classifying Space* of a group G, denoted by BG, this is a topological space whose the fundamental group is isomorphic to G, such that G acts freely on KG, using this construction one can define the K-groups, the central objects of the algebraic K-theory. In this course, we plain to:

- Prove the Seifert-Van Kampen Theorem for CW-complexes;
- Construct the Universal Covering Spaces for CW-complexes;
- Construct the so called Classifying space;
- Present briefly the higher homotopy groups and present the definition of K-groups.

Prerequisites

Basic notions of general topology and group theory. While a first course in Algebraic Topology is highly beneficial, it is not a formal prerequisite, as the necessary definitions and fundamental results will be presented for the particular case of CW-complexes.

Target Audience

Graduate students and last year undergrads.

Schedule of the Course

Class I (Basic Definitions)

- Present brief history of Algebraic Topology.

- Define the CW-complex.
- Present some examples of CW-complex: Torus, Spheres, Projective Space, some Lie groups.

Class II (CW-complex homology I)

- Prove some properties about CW-complexes: Shows that every CW-complex is a Hausdorff Space.
- Present the singular homology for Topological space.
- Define the CW-complex.
- Shows that Singular homology coincides with the CW-complex.
- Present the Eilenberg-Stenrood Axioms.

Class III (CW-complex Homology II)

- Compute the homology of classical CW-complexes.
- Excision property.
- Present the incidence number.
- Present the Mayer-Vietories long exact Sequence.

Class IV (The Fundamental Group)

- Define the topological fundamental group and the combinatorial one.
- Shows that the two definitions of fundamental groups coincides for CW-complexes.
- Give examples of fundamental groups of CW-complexes.
- Present and prove the Hurewicz theorem for π_1.

Class V (Covering Spaces)

- Present the topological definition of covering spaces and give examples.
- Shows that the covering space of a CW-complex is also a CW-complex.
- Construct some examples.
- Proves that every CW-complex has a universal cover.
- Give examples of universal coverings.

Class VI (More about covering spaces)

- Present the so called Galois correspondence for covering spaces.

Class VII (The Classifying Space)

- Present the definition of the Classifying space.
- Construct the space BG.
- Give some examples of the classifying space.

Class VIII (Higher homotopy groups)

- Define the higher homotopy groups.
- Present (without proof) the complete Hurewicz Theorem.
- Define the K-groups and the K-theory according Quillen.

References

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