Pólya's Urns with hypergraph-based interactions

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Abstract

Given a finite connected hypergraph H, place a bin at each vertex. At discrete times, a ball is added to each hyperedge. In a hyperedge, one of the bins gets the ball with probability proportional to its current number of balls. This model was first introduced by [BBCL15]. The convergence in the graph-based case was proved in [BBCL15; CL14; Lim16], and, in this report, we prove a similar result about the convergence of this process for the hypergraph-based case. This report was written to summarize our results that were made on the event "Jornadas de Pesquisa" that took place at ICMC-USP between 04/01/2023 and 10/02/2023.

1 Introduction

The classical Pólya's urn is a sampling model that consists of an urn with balls of two colors, e.g. blue and red. At each step, one ball is chosen at random from the urn and then replaced at the urn together with a ball of the same color. We can reinterpret the model by having two urns, one for each color, and adding balls to the pair of urns at each step, choosing where the new balls go with probabilities proportional to the current number of balls of each color. There are classical results relating to the almost sure convergence and distribution of the proportions of balls of each color.

In [BBCL15], the authors considered a variation of the classical Pólya's urn, called graph-based Pólya's urns and hypergraph-based Pólya's urns. Let $H = (V, E)$ be a finite connected hypergraph with $V = [m] = \{1, \dots, m\}$ and $|E| = N$, and assume that there is at each vertex i a bin with $B_i(0) \geq 1$ balls. Consider a random process consisting of adding N balls to these bins according to the following rule: if the number of balls after step $n-1$ at each vertex is $B_1(n-1), \dots, B_m(n-1)$, step n consists in adding, at each hyperedge I, one ball for a vertex of I in a way that the probability of the ball being added to the vertex i is

$$\mathbb{P}[i \text{ is chosen in } I \text{ at step } n] = \frac{B_i(n-1)}{\sum_{j \in I} B_j(n-1)}.$$

Let $N_0 = \sum_{i=1}^{m} B_i(0)$ be the initial number of balls and

$$x_i(n) = \frac{B_i(n)}{N_0 + nN}, \, i \in [m],$$

be the proportion of balls at the vertex i after n steps, then define

$$x(n) = (x_1(n), \dots, x_m(n)).$$

For graphs, when $|I| = 2$ for all $I \in E$, the results of [BBCL15], [CL14] and [Lim16] can be summarized as follows:

Theorem. *Let G be a finite, connected graph.*

(i) If G is not balanced bipartite, then there's a unique point $v = v(G)$ such that $x(n)$ converges to v almost surely.

(ii) If G is balanced bipartite, then there's a interval $\mathcal{J} = \mathcal{J}(G)$ such that $x(n)$ converges to a point $v \in \mathcal{J}$ almost surely.

The approach to proving this theorem remains in the fact that sequence $x(n)$ is a stochastic algorithm approximation. This allows us to see $x(n)$ as small perturbations of a vector field F, and relate the behavior of F to that of $x(n)$. More details about this algorithm and the vector field F will be given in Section 2. Our approach is basically the same, since this is also true for hypergraphs, and, with some adaptations and new ideas to generalize the arguments, we prove a similar theorem for the case of hypergraph-based Pólya-urns.

Given a hypergraph H, and a ordering of the hyperedges $\{I_1, \dots, I_N\}$ in E, the incidence matrix $I(H) = (a_{ij})$ of H is the $N \times m$ matrix, where $a_{ij} = 1$ if $j \in I_i$, and 0, otherwise. Besides that, we define $\Gamma = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_1 + \cdots + x_m = 0\}$, that is the tangent space with respect to the $(m+1)$-dimensional simplex $\Delta = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_1 + \cdots + x_m = 1\}$.

Our main result is the following

Main Theorem. *Let H be a finite and connected hypergraph.*

(i) If $I(H)|_\Gamma$ is injective, then there is a unique deterministic point $v = v(H)$ such that $x(n)$ converges to v almost surely.

(ii) If $I(H)|_\Gamma$ is not injective and $x(n)$ doesn't converge to $\partial\Delta$, then there is a closed subset of Δ, $\mathcal{J} = \mathcal{J}(H)$, such that $x(n)$ converges to a point of $\mathcal{J}$ almost surely.

We denote by $\partial\Delta \subset \Delta$ the set of the points with at least one coordinate equal to 0. Let $int(\Delta) = \Delta \setminus \partial\Delta$. As is the case for graphs, in some cases $\mathcal{J}$ can be a singleton and the condition of the existence of an equilibrium in $int(\Delta)$ is not necessary. But when $\mathcal{J}$ is not a singleton and $x(n)$ doesn't converge to $\partial\Delta$, the point to which $x(n)$ converges depends on the realizations of the process.

As for the classical and graph-based Pólya urns, hypergraph-based Pólya urns can be used to model even more complex situations. In [BBCL15], they mention a model of competing networks that can be generalized for the hypergraph case: Imagine that are 4 companies, denoted by letters A, G, M, and S. The company A sells OS, SP, and NB, M sells OS, SE, and NB, G sells SP, SE, and NB, and S sells SP and NB. The natural question to be made is which company will sell more products? This can be modeled using a hypergraph representation, where each product is a vertice, and the hyperedges represent products that the company sells. In broad strokes, with some other simplifications, this model describe the long term behavior of such companies.

2 Stochastic Approximation Algorithm

A stochastic approximation algorithm is a discrete time stochastic process whose general form can be written as

$$x(n+1) - x(n) = \gamma_n H(x(n), \xi(n)),$$

where H is a measurable function that characterizes the algorithm, $\{x(n)\}_{n\geq 0}$ is the sequence of parameters to be recursively updated, $\{\xi(n)\}_{n\geq 0}$ is a sequence of random inputs where $H(x(n), \xi(n))$ is observable, and $\{\gamma_n\}_{n\geq 0}$ is a sequence of small (in some sense) nonnegative scalar gains.

The dynamical approach is a method to analyze stochastic approximation algorithms. It says that the recursive expression can be related to the autonomous ODE

$$\frac{dx(t)}{dt} = \overline{H}(x(t)),$$

where $\overline{H}(x) = \lim_{n\to\infty} \mathbb{E}[H(x, \xi(n))]$.

We will show that we can write the algorithm as

$$x(n+1) - x(n) = \gamma_n [F(x(n)) + u_n],$$

where u_n is a random variable related to $\xi(n)$ with 0 expectancy. Thus, the ODE will become

$$\frac{dx(t)}{dt} = F(x(t)),$$

which is easier to analyze.

Let $\mathscr{F}_n$ be the σ-algebra generated by the process until step n and $C_i(n)$ be the number of balls added to vertex i at step n. Consider the random variables $\delta_{I\to i}(n+1) \in \{0,1\}$ for $i \in I$ such that $\sum_{i\in I} \delta_{I\to i}(n+1) = 1$ and

$$\mathbb{E}[\delta_{I\to i}(n+1)|\mathscr{F}_n] = \frac{x_i(n)}{x_I(n)}.$$

Assume that $\delta_{I_1\to i_1}(n+1)$ and $\delta_{I_2\to i_2}(n+1)$ are independent for every $I_1 \neq I_2$. Then $C_i(n+1) = \sum_{I\in E^i} \delta_{I\to i}(n+1)$. This way, we have that

$$\begin{aligned}
x_i(n+1) - x_i(n) &= \frac{B_i(n) + C_i(n+1)}{N_0 + (n+1)N} - \frac{B_i(n)}{N_0 + nN} \\
&= \frac{B_i(n)}{N_0 + (n+1)N}\left(1 - \frac{N_0 + (n+1)N}{N_0 + nN}\right) + \frac{C_i(n+1)}{N_0 + (n+1)N} \\
&= \frac{1}{N_0 + (n+1)N}\left(-\frac{B_i(n)N}{N_0 + nN}\right) + \frac{C_i(n+1)}{N_0 + (n+1)N} \\
&= \frac{1}{\frac{N_0}{N} + (n+1)}\left(-x_i(n) + \frac{1}{N}C_i(n+1)\right).
\end{aligned}$$

If we write $\gamma_n = \frac{1}{\frac{N_0}{N}+(n+1)}$ and $\xi_i(n) = \frac{1}{N}C_i(n+1)$, then we can define $\xi(n) = (\xi_1(n), \ldots, \xi_m(n))$ and we will have

$$x(n+1) - x(n) = \gamma_n(-x(n) + \xi(n)).$$

Given a hyperedge $I \in E$, let $x_I = \sum_{i \in I} x_i$, and define $E^i = \{I \in E : i \in I\}$. Consider the random variable $u_n = \xi(n) - \mathbb{E}[\xi(n)|\mathscr{F}_n]$ and notice that

$$\begin{aligned}
\mathbb{E}[\xi_i(n)|\mathscr{F}_n] &= \mathbb{E}[\frac{1}{N}C_i(n+1)|\mathscr{F}_n] \\
&= \frac{1}{N}\mathbb{E}[C_i(n+1)|\mathscr{F}_n] \\
&= \frac{1}{N}\mathbb{E}\left[\sum_{I \in E^i} \delta_{I \to i}(n+1)|\mathscr{F}_n\right] \\
&= \frac{1}{N}\sum_{I \in E^i} \mathbb{E}[\delta_{I \to i}(n+1)|\mathscr{F}_n] \\
&= \frac{1}{N}\sum_{I \in E^i} \frac{x_i(n)}{x_I(n)}.
\end{aligned}$$

Define the vector field $F : \Delta \to \mathbb{R}^m$, $F(x(n)) = (F_1(x(n)), F_2((n)x), \ldots, F_m((n)x))$, where $F_i(x(n)) = -x_i(n) + \frac{1}{N}\sum_{I \in E^i} \frac{1}{x_I(n)}$. Now we can write

$$x(n+1) - x(n) = \gamma_n[F(x(n)) + u_n],$$

where u_n is a random variable with conditional expectation $\mathbb{E}[u_n|\mathscr{F}_n] = 0$.

We will specify the domain Δ of F. Fix $c < \dfrac{1}{N}$ and let $\Delta \subseteq R^m$ be the set of the m-tuples $(x_1, \ldots, x_m)$ that satisfy:

(i) $x_i \geq 0$ and $\sum_{i=1}^m x_i = 1$.

(ii) $x_I \geq c$ for all $I \in E$.

We can show that Δ is positively invariant under the ODE $\frac{dv(t)}{dt} = F(v(t))$. Consider $I \in E$ a hyperedge of H,

$$\begin{aligned}
\frac{d}{dt}v_I &= \sum_{i \in I}\left(-v_i + \frac{1}{N}\sum_{J \in E^i}\frac{v_i}{v_J}\right) \\
&\geq -v_I + \frac{1}{N}\sum_{i \in I}\frac{v_i}{v_I} \\
&= -v_I + \frac{1}{N}
\end{aligned}$$

Now, if v is a point on the boundary of Δ, then there must exist some $I \in E$ such that $v_I = c$. So

$$\frac{d}{dt}v_I = -c + \frac{1}{N} > 0$$

and we may conclude that Δ is positively invariant under that ODE.

Definition. *Let $U \subset \Delta$ be a closed set, and let $F : U \to \mathbb{R}_m$ be a continuous vector field with unique integral curves, then:*

i) A point $x \in U$ is an equilibrium point if $F(x) = 0$. x is called a stable equilibrium if all eigenvalues of $JF(x)$ have negative real part, and an unstable equilibrium point if one of the eigenvalues of $JF(x)$ have positive real part. We denote the set of all equilibrium points by Λ and call it the equilibria set of F.

ii) A strict Lyapunov function for F is a continuous map $L : U \to \mathbb{R}$ which is strictly monotone along any integral curve of F outside of Λ. If F has a strict Lyapunov function, we call F gradient-like.

Consider the function $L : \Delta \to \mathbb{R}$, which is given by

$$L(v_1, \dots, v_m) = -\sum_{i=1}^{m} v_i + \frac{1}{N} \sum_{I \in E} \log v_I$$

Lemma 2.1. *L is a strict Lyapunov function F.*

Proof. Calculating the derivative of L with respect to v_i, we have

$$\frac{\partial L}{\partial v_i} = -1 + \frac{1}{N} \sum_{I \in E^i} \frac{1}{v_I},$$

thus

$$\frac{dv_i}{dt} = F_i(v(t)) = v_i \left(-1 + \frac{1}{N} \sum_{I \in E^i} \frac{1}{v_I} \right) = v_i \frac{\partial L}{\partial v_i}.$$

Now consider an integral curve of F given by $v = (v_1(t), \dots, v_m(t)), t \geq 0$, then

$$\frac{d}{dt}(L \circ v) = \sum_{i=1}^{m} \frac{\partial L}{\partial v_i} \frac{dv_i}{dt} = \sum_{i=1}^{m} v_i \left(\frac{\partial L}{\partial v_i} \right)^2 \geq 0.$$

Note that equality holds in the last expression if, and only if, $v_i \left(\frac{\partial L}{\partial v_i} \right)^2 = 0$ for every $i \in [m]$ wich is equivalent to $F(v) = 0$. This proves that L is a strict Lyapunov function for F. □

3 Limit set theorem

For our purposes, the results in [Ben96] can be summarized as follows.

Theorem 3.1. *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous gradient-like vector field with unique integral curves, let Λ be its equilibria set, let L be a strict Lyapunov function, and let $\{x(n)\}_{n \geq 0}$ be a solution to the recursion*

$$x(n+1) - x(n) = \gamma_n [F(x(n)) + u_n],$$

where γ_n is a decreasing sequence satisfying $\sum_{n \geq 0} \gamma_n = \infty$, $\lim_{n \to \infty} \gamma_n = 0$ and $\{u_n\}_{n \geq 0} \subset \mathbb{R}^m$. Assume that

1. $\{x_n\}_{n\geq 0}$ *is bounded,*

2. *for each* $T > 0$,
$$\lim_{n\to\infty}\left(\sup_{\{k:0\leq\tau_k-\tau_n\leq T\}}\left\|\sum_{i=n}^{k-1}\gamma_i u_i\right\|\right)=0,$$
where $\tau_n = \sum_{i=0}^{n-1}\gamma_i$, *and*

3. $L(\Lambda)\subset R$ *has empty interior.*

Then the limit set of $\{x_n\}_{n\geq 0}$ *is a connected subset of* Λ.

We will now check that our random process satisfies all 3 conditions. Clearly, $\{x_n\}_{n\geq 0}$ is bounded and γ_n is a decreasing sequence satisfying $\sum_{n\geq 0}\gamma_n = \infty$, $\lim_{n\to\infty}\gamma_n = 0$. Let
$$M_n = \sum_{i=0}^{n-1} y_i u_i$$
$\{M_n\}_{n\geq 0}$ is a Martingale adapted to the filtration $\{\mathscr{F}_n\}_{n\geq 0}$:
$$\mathbb{E}[M_{n+1}|\mathscr{F}_n] = \sum_{i=0}^{n-1} y_i u_i + \mathbb{E}[\gamma_n u_n|\mathscr{F}_n] = \sum_{i=0}^{n-1} y_i u_i = M_n.$$
Furthermore, because for any $n\geq 0$
$$\sum_{i=0}^{n}\mathbb{E}[\|M_{i+1}-M_i\|^2|\mathscr{F}_i] \leq \sum_{i=0}^{n}\gamma_i^2 \leq \sum_{i\geq 0}\gamma_i^2 < \infty \text{ a.s.},$$
the sequence $\{M_n\}_{n\geq 0}$ converges almost surely to a finite random vector. In particular, it is a Cauchy sequence and so condition 2 holds almost surely. It remains to check condition 3.
For each $S\subset[m]$, let
$$\Delta_S = \{v\in\Delta : v_i = 0 \text{ iff } i\notin S\}$$
denote the face of Δ determined by S. Δ_S is a manifold with corners, positively invariant under the ODE.

Definition. $v\in\Delta_S$ *is an* $S-$*singularity for* L *if*
$$\frac{\partial L}{\partial v_i}(v) = 0 \textit{ for all } i\in S.$$

Let Λ_S denote the set of $S-$singularities for L. It's easy to show that $\Lambda = \bigcup_{S\subset[m]}\Lambda_S$ and that $L|_{\Delta_S}$ is a C^∞ function. Thus by Sard's theorem $L(\Lambda_S)$ has zero Lebesgue measure, so $L(\Lambda)$ has zero Lebesgue measure as well. In particular, it has empty interior.

4 Non-convergence to unstable equilibria

The following lemmas are proven for graph-based Pólya's urns in [BBCL15] and generalize with small changes to the hypergraph case. We give their full proofs for completeness.

Lemma 4.1. *Let H be a finite, connected hypergraph and let $v \in \Lambda$. The following are equivalent:*

1. *v is an unstable equilibrium.*
2. *There is $i \in [m]$ such that $v_i = 0$ and $\frac{\partial L}{\partial v_i}(v) > 0$.*

Proof. Consider the jacobian matrix $JF(v) = \left(\frac{\partial F_i}{\partial v_j}(v)\right)$. A simple calculation gives us

$$\frac{\partial F_i}{\partial v_j}(v) = \begin{cases} v_i \frac{\partial^2 L}{\partial v_i \partial v_j}(v) & \text{if } E^i \cap E^j \neq \emptyset \text{ and } i \neq j, \\ \frac{\partial L}{\partial v_i}(v) + v_i \frac{\partial^2 L}{\partial v_i^2}(v) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, assume that $v \in \Delta_S$ with $S = \{1, \dots, k\} \subseteq [m]$. Thus

$$JF(v) = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

where $B = \operatorname{diag}\left(\frac{\partial L}{\partial v_{k+1}}(v), \dots, \frac{\partial L}{\partial v_m}(v)\right)$. The spectrum of $JF(v)$ is then the union of the spectra of A and B.
Define the inner product $(x, y) = \sum_{i=1}^{k} \frac{x_i y_i}{v_i}$. It's easy to check that A is self-adjoint with respect to that inner product and $(Ax, y) = \langle Dx, y \rangle$, where D is the Hessian matrix of L restricted to the first k coordinates. Thus, the eigenvalues of A are all real and non-positive (by the concavity of L). For that reason, $JF(v)$ has a real positive eigenvalue if and only if $\frac{\partial L}{\partial v_i}(v)$ for some $i \in S \setminus [m]$. $\square$

Lemma 4.2. *Let $v \in \Delta$ with $v_1 = 0$ and $\partial L / \partial v_1(v) > 3\delta$. Then there exists a neighborhood $\mathcal{N}$ of v, an element $u \in \mathcal{N}$ and $\epsilon_0 > 0$ such that*

1. *$\partial L / \partial v_1(u) > 3\delta + \frac{|E^1|\epsilon_0}{N}$*
2. *for all $w \in \mathcal{N}$ and $I \in E^1$ it holds*

$$\frac{1}{w_I} > \frac{1}{u_I} - \epsilon_0$$

Proof. Fix $\epsilon_0 \in \left(0, \left(\frac{\partial L}{\partial v_1}(v) - 3\delta\right) \frac{N}{|E^1|}\right)$, so that $3\delta + \frac{|E^1|\epsilon_0}{N} < \frac{\partial L}{\partial v_1}(v)$. Because $\partial L / \partial v_1$ is continuous, we can fix a neighborhood $\mathcal{N}$ of v satisfying condition 1.

Each $w \in \overline{\mathcal{N}} \mapsto 1/w_I$, $I \in E^1$ is uniformly continuous, so there exists $\overline{\delta}$ such that $|w - \tilde{w}| < \overline{\delta} \implies \left|\frac{1}{w_I} - \frac{1}{\tilde{w}_I}\right| < \epsilon_0$ for all $w, \tilde{w} \in \overline{\mathcal{N}}$ and $I \in E^1$. Thus, if we make $\operatorname{diam}(\mathcal{N}) < \overline{\delta}$ condition 2 is also satisfied. □

Lemma 4.3. *Let H be a finite, connected hypergraph. Let $v \in \Lambda$ with $v_1 = 0$. If $\partial L/\partial v_1(v) > 0$, then*

$$\mathbb{P}\left[\lim_{n\to\infty} x(n) = v\right] = 0$$

Proof. We claim that

$$\mathbb{P}\left[\lim_{n\to\infty} B_1(n) = \infty\right] = 1.$$

Fix some $I \in E^1$ and let Z_n be the event that 1 is chosen in I at step $n+1$. Because $1 \le B_i(n) \le N_0 + nN$ for all $i \in [m]$, it follows that

$$\mathbb{P}\left[Z_n | Z_k^{\mathsf{c}}, \dots, Z_{n-1}^{\mathsf{c}}\right] \ge \frac{1}{|I|(N_0 + nN)}$$

for every $I \in E^1$ and $k < n$. The claim follows by an adaptation of the Borel-Cantelli lemma.
Let $\delta > 0$ and $\mathcal{N}$ as in Lemma 4.2 and fix $B > 0$ large enough (to be specified later), and define

$$\mathcal{Y}_n = \{x(k) \in \mathcal{N}, \forall k \ge n\} \cap \{B_1(n) > B\},\ n > 0.$$

By the previous claim, $\{\lim_{j\to\infty} x(j) = v\} \subset \bigcup_{m\ge n} \mathcal{Y}_m$ for all $n > 0$. Thus it is enough to show that

$$\mathbb{P}[\mathcal{Y}_n] = 0 \text{ for sufficiently large } n.$$

For a fixed n_0, let $\mathcal{G}_n = \mathcal{F}_n \cap \mathcal{Y}_{n_0}$, and let $c > 0$ such that

$$\left[1 + \frac{\delta(1+2\delta)}{1 + \frac{3}{2}\delta}\right] = 1 + c$$

We claim that if B is large enough, then there is $n_0 > 0$ such that

$$\mathbb{E}[\log x_1((1+\delta)n) | \mathcal{G}_n] \ge \log x_1(n) + \frac{1}{2}\log(1+c) \text{ for all } n > n_0.$$

Let $t \in \{n+1, \dots, (1+\delta)n\}$. Restricted to $\mathcal{Y}_{n_0}$, we have

$$\begin{aligned}
\mathbb{P}[1 \text{ is chosen in } I \text{ at step } t] &= \frac{B_1(t-1)}{B_I(t-1)} \\
&\ge \frac{B_1(n)}{N_0 + (t-1)N} \frac{1}{x_I(t-1)} \\
&\ge \frac{B_1(n)}{N_0 + (t-1)N} \left(\frac{1}{u_I} - \epsilon_0\right).
\end{aligned}$$

Define a family of independent Bernoulli random variables $\{E_{t,I}\}$, $t = n+1, \ldots, (1+\delta)n$, $I \in E^1$ such that

$$\mathbb{P}[E_{t,I} = 1] = \frac{B_1(n)}{N_0 + (t-1)N} \left(\frac{1}{u_I} - \epsilon_0 \right).$$

Now couple $\{E_{t,I}\}$ to our model as follows: if $E_{t,I} = 1$, then 1 is chosen in I at step t. Then

$$\begin{aligned}
\mathbb{E}\left[\sum_{\substack{n+1 \le t \le (1+\delta)n \\ I \in E^1}} E_{t,I} \right] &= \sum_{I \in E^1} \left(\sum_{t=n+1}^{(1+\delta)n} \frac{B_1(n)}{N_0 + (t-1)N} \left(\frac{1}{u_I} - \epsilon_0 \right) \right) \\
&= B_1(n) \left[\sum_{I \in E^1} \left(\frac{1}{u_I} - \epsilon_0 \right) \right] \sum_{t=n+1}^{(1+\delta)n} \left(\frac{1}{N_0 + (t-1)N} \right) \\
&\ge B_1(n) \left[\frac{1}{N} \sum_{I \in E^1} \left(\frac{1}{u_I} - \epsilon_0 \right) \right] \log \left(1 + \frac{\delta n N}{N_0 + nN} \right) \\
&\ge B_1(n)(1+3\delta) \log \left(1 + \frac{\delta n N}{N_0 + nN} \right).
\end{aligned}$$

If n_0 is large enough (such that $\frac{n_0 N}{N_0 + n_0 N} > \frac{1}{1+\frac{1}{2}\delta}$), we get

$$\mathbb{E}\left[\sum_{\substack{n+1 \le t \le (1+\delta)n \\ I \in E^1}} E_{t,I} \right] \ge B_1(n) \frac{\delta(1+3\delta)}{1+\frac{3}{2}\delta}.$$

By Chernoff bounds, if $\epsilon_1 > 0$ then there is B_0 large enough such that

$$\mathbb{P}\left[\sum_{\substack{n+1 \le t \le (1+\delta)n \\ I \in E^1}} E_{t,I} > B_1(n) \frac{\delta(1+2\delta)}{1+\frac{3}{2}\delta} \right] > 1 - \epsilon_1$$

for every $B_1(n) > B_0$. Whenever the previous event holds, the coupling gives that

$$B_1((1+\delta)n) - B_1(n) \ge \sum_{\substack{n+1 \le t \le (1+\delta)n \\ I \in E^1}} E_{t,I} > B_1(n) \frac{\delta(1+2\delta)}{1+\frac{3}{2}\delta}, \text{ thus}$$

$$x_1((1+\delta)n) > x_1(n)(1+c).$$

Accordingly, $\mathbb{P}[x_1((1+\delta)n) > x_1(n)(1+c) | \mathcal{G}_n] > 1 - \epsilon_1$. Now, because $x_1((1+\delta)n) > \frac{x_1(n)}{1+\delta}$ and ϵ_1 can be taken arbitrarily small, we get

$$\begin{aligned}
\mathbb{E}[\log x_1((1+\delta)n) | \mathcal{G}_n] &> (1-\epsilon_1) \log(x_1(n)(1+c)) + \epsilon_1 \log \left(\frac{x_1(n)}{1+\delta} \right) \\
&> \log x_1(n) + \frac{1}{2} \log(1+c),
\end{aligned}$$

thus proving the claim. To conclude the proof of the lemma, we assume by contradiction that $\mathbb{P}[\mathcal{Y}_n] > 0$ for some $n > n_0$. Define $T_k = (1+\delta)^k n$ and $X_k = log x_1(T_k)$. Then

$$\mathbb{E}\left[X_{k+1}|\mathcal{G}_n\right] = \mathbb{E}\left[\mathbb{E}\left[X_{k+1}|\mathcal{G}_{T_k}\right]|\mathcal{G}_n\right] \geq \mathbb{E}\left[X_k|\mathcal{G}\right] + \frac{1}{2}\log(1+c).$$

By induction,

$$\mathbb{E}\left[X_k|\mathcal{G}_n\right] \geq X_0 + \frac{k}{2}\log(1+c) \geq -\log(N_0+nN) + \frac{k}{2}\log(1+c)$$

which is a contradiction, because the left hand side is bounded. $\square$

Now, let v be an unstable equilibrium. We can assume $v_1 = 0$ and $\frac{\partial L}{\partial v_1}(v) > 0$, otherwise we just reorder the vertices of the hypergraph. Thus, by Lemma 4.3, almost surely $x(n)$ does not converge to v.

5 Proof of Main Theorem

This section will be devoted to the proof of the main theorem of this report:

Main Theorem. *Let H be a finite and connected hypergraph.*

(i) If $I(H)|_\Gamma$ is injective, then there is a unique deterministic point $v = v(H)$ such that $x(n)$ converges to v almost surely.

(ii) If $I(H)|_\Gamma$ is not injective and $x(n)$ doesn't converge to $\partial\Delta$, then there is a closed subset of Δ, $\mathcal{J} = \mathcal{J}(H)$, such that $x(n)$ converges to a point of $\mathcal{J}$ almost surely.

The proof was separated into several lemmas and many of the ideas used here were the same as in [CL14] and [Lim16] with some modifications and new insights.

Given $w \in \Delta_S$, and $\chi \in (0, \min_{i\in S} w_i]$, let $\Delta^{w,\chi} = \{v \in \Delta : v_i \geq \chi, \forall i \in S\}$. By Theorem 3.1 and Lemma 4.3, there is a non-unstable equilibrium w.

Lemma 5.1. *Let w be a non-unstable equilibrium. Then there's a closed subset of Δ, $J = J(w,\chi)$, such that the orbit of $F|_{\Delta^{w,\chi}}$ converges to J.*

Proof. Let $H : \Delta^{w,\chi} \to \mathbb{R}$ be a Lyapunov function defined by $H(v) = \sum_{i\in S} w_i \log(v_i)$. Let $c^0 = \sum_{i\in S} w_i log(v_i(0))$, and, since $H(v) \leq 0$, consider the set $H^-1[c^0, 0] = \{v \in \Delta^{w,\chi} : H(v) \geq c^0\}$. For $\chi > 0$ small enough, $H^-1[c^0, 0] \subset \Delta^{w,\chi}$, and thus H is differentiable in $\Delta^{w,\chi}$.

Taking the derivate of H along the orbit of v

$$\frac{d}{dt}(H \circ v) = \sum_{i\in S} w_i \frac{1}{v_i}\frac{dv_i}{dt} = \sum_{i\in S} w_i \frac{\partial L}{\partial v_i} = \sum_{i=1}^{m} w_i \frac{\partial L}{\partial v_i} = -1 + \frac{1}{N}\sum_{I\in E}\frac{w_I}{v_I}. \qquad (1)$$

Let $f : \Delta^{w,\chi} \to \mathbb{R}$ as $f(v) = -1 + \frac{1}{N}\sum_{I\in E^i}\frac{w_I}{v_I}$. Note that $f(w) = 0$. We claim that w is a global minimum of f. Hence, because $x > 0 \to \frac{1}{x}$ is convex, and f is a sum of convex functions, f is convex, and thus it is sufficient to show that w is a local minimum of f.

Let $v = w + \epsilon \in \Delta$ with $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ and $\|\epsilon\|$ sufficiently small. Notice that as $w + \epsilon \in \Delta$, then $\epsilon \in \Gamma = \left\{x \in \mathbb{R}^m : \sum_{i=0}^m x_i = 0\right\}$, also if $i \notin S$, then $\epsilon_i \geq 0$, since $v_i = e_i \geq 0$ for $i \notin S$. To prove our claim, we will use the following inequality

$$\frac{x}{x+\epsilon} - 1 \geq -\frac{\epsilon}{x}, \forall x > 0, \epsilon > -x,$$

that holds equality if, and only if, $\epsilon = 0$. Applying the inequality,

$$\begin{aligned}
f(v) - f(w) &= -1 + \frac{1}{N} \sum_{I \in E} \frac{w_I}{v_I} \\
&= \frac{1}{N} \sum_{I \in E} \left(\frac{w_I}{w_I + \epsilon_I} - 1 \right) \\
&\geq \frac{1}{N} \sum_{I \in E} -\frac{\epsilon_I}{w_I} \\
&= -\sum_{i=0}^{m} \epsilon_i \left(\frac{1}{N} \sum_{I \in E^i} \frac{1}{w_I} \right) \\
&= -\sum_{i=0}^{m} \epsilon_i \left(1 + \frac{\partial L}{\partial v_i}(w) \right) \\
&= -\sum_{i=0}^{m} \epsilon_i \frac{\partial L}{\partial v_i}(w) \\
&= -\sum_{i \in S} \epsilon_i \frac{\partial L}{\partial v_i}(w) - \sum_{i \notin S} \epsilon_i \frac{\partial L}{\partial v_i}(w) \\
&\geq 0
\end{aligned}$$

since $\epsilon_i \frac{\partial L}{\partial v_i} = 0$ for $i \in S$, and $\epsilon_i \frac{\partial L}{\partial v_i} \leq 0$ for $i \notin S$. By the inequality, the equality holds if, and only if, $\epsilon_I = 0$ for all $I \in E$. This system is equivalent to $I(H)\epsilon = 0$ for $\epsilon \in \Gamma = \left\{x \in \mathbb{R}^m : \sum_{i=1}^m x_i = 0\right\}$. We consider the set

$$J(w, \chi) = \left\{v \in \Delta^{w,\chi} : I(H)(v - w) = 0\right\}. \tag{2}$$

Notice that J is a closed set, since it's the pre-image of a closed set by a continuous function, and $f|_J \equiv 0$. □

If $I(H)|_\Gamma$ is injective, by (2), $J(w, \chi)$ must be a unique point. Let $v \in int(\Delta)$. For χ small enough, $v \in \Delta^{w,\chi}$, and the orbit of $F|_{\Delta^{w,\chi}}$ converges to w, which concludes the proof of part (a).

Now we prove part (b). If $I(H)|_\Gamma$ is not injective, let $\mathcal{J} = \mathcal{J}(w)$ be the maximal extension of $J(w, \chi)$ on Δ. We have that $\mathcal{J}$ has the following properties:

- It is an affine space $w + \ker(I(H))$ restricted to Δ.

- $\frac{\partial L}{\partial v_i}\Big|_{\mathcal{J}}$ is constant and equal to $\frac{\partial L}{\partial v_i}(w)$ for all i.

In fact, it doesn't depend on the choice of w.

Lemma 5.2. *The space $\mathcal{J}$ is the set of all non-unstable equilibria.*

Proof. Let w and $\widetilde{w}$ be two non-unstable equilibria. There is a $\chi > 0$ sufficiently small such that $\Delta^{w,\chi} \cap \Delta^{\widetilde{w},\chi} \neq \emptyset$. Every orbit of F starting from $\Delta^{w,\chi} \cap \Delta^{\widetilde{w},\chi}$ converges to both $\mathcal{J}(w)$ and $\mathcal{J}(\widetilde{w})$, so $\mathcal{J}(w) \cap \mathcal{J}(\widetilde{w}) \neq \emptyset$. Since it's a translation of the same subspace that intersects in a point, it must be the same space. Thus $\mathcal{J} = \mathcal{J}(w) = \mathcal{J}(\widetilde{w})$.

Let $w \in \Delta$ be a non-unstable equilibria. By the properties of $\mathcal{J}$, $\frac{\partial L}{\partial v_i}|_{\mathcal{J}} = \frac{\partial L}{\partial v_i}(w) \leq 0$ for all i. By Lemma 4.1, $\mathcal{J}$ is the set of all non-unstable equilibria. □

Let $int(\mathcal{J})$ denote the interior of $\mathcal{J}$ by viewing it as immersed in the euclidean space of dimension $\dim(\ker(I(H))$. If $x(n)$ doesn't converge to $\partial\Delta$, then, by Theorem 3.1 and Lemma 4.3, we have that $\Lambda_{[m]} \neq \emptyset$. This guarantees that $int(\mathcal{J}) \subset int(\Delta)$, since $\Lambda_{[m]} \subset \mathcal{J}$ and $\mathcal{J}$ is not fully contained in a face of Δ.

Lemma 5.3. *Suppose that $\Lambda_{[m]} \neq \emptyset$ and let $w \in int(\mathcal{J})$, then all eigenvalues of $DF(w)$ are real, and any eigenvalues in a transverse direction to $\mathcal{J}$ is negative.*

Proof. Since w is a non-unstable equilibrium, all eigenvalues of $DF(w)$ are non-positive, and it's sufficient to show that $\ker(DF(w)) = \ker(I(H))$.

Let $v \in \ker(I(H))|_{\Gamma}$. As $w \in int(\mathcal{J})$, $w + tv$ is also a non-unstable equilibrium. By the Taylor expansion of F at $w + tv$:

$$F(w + tv) = F(w) + DF(w)(tv) + R(t)$$
$$DF(w)v = -\frac{R(t)}{t} \xrightarrow[t\to 0]{} 0$$
$$DF(w)v = 0$$

Thus $v \in \ker(DF(w))$, and $\ker(I(H)|_{\Gamma}) \subset \ker(DF(w))$.

By Lemma 4.1, for $w \in \Lambda_{[m]}$, $DF(w) : T_w\Delta \to T_w\Delta$ is equal to the matrix $C = \left(v_i \frac{\partial L}{\partial v_i \partial v_j}\right)$ restricted to $T_w\Delta$. Let $A = \left(\frac{\partial L}{\partial v_i \partial v_j}\right)$ be Hessian matrix of L at coordinates $v_1, \ldots, v_m$. Since the rows of C are nonzero multiples of the rows of A, $\operatorname{rank}(C) = \operatorname{rank}(A)$.

Let B be the incidence matrix of H with each row multiplied by $\frac{1}{w_I}$, where I is the hyperedge that denotes said row. Notice that $\operatorname{rank}(I(H)) = \operatorname{rank}(B)$. We claim that $-\frac{1}{N}B^T B = A$. In fact, the ith-row of B^T is $\left(\delta_{I_1}(i)\frac{1}{v_{I_1}}, \ldots, \delta_{I_N}(i)\frac{1}{v_{I_N}}\right)$, where $\delta_{I_k}(i) = 0$ if $i \in I_k$, and 0, otherwise. And jth-column of B is $\left(\delta_{I_1}(j)\frac{1}{v_{I_1}}, \ldots, \delta_{I_N}(j)\frac{1}{v_{I_N}}\right)$. Multiplying the ith-row of B^T with the jth-column of B is equal to

$$\sum_{I \in E^i \cap E^j} \frac{1}{(v_I)^2}.$$

Calculating $\dfrac{\partial L}{\partial v_i \partial v_j}$, we obtain

$$-\frac{1}{N}\sum_{I\in E^i\cap E^j}\frac{1}{(v_I)^2},$$

which proves our claim. Hence $\mathrm{rank}(B^T B) = \mathrm{rank}(A)$.

As $\mathrm{rank}(B^T B) = \mathrm{rank}(B)$, we have that $\mathrm{rank}(I(H)|_\Gamma) = \mathrm{rank}(A)$, which implies that $\mathrm{rank}(I(H)|_\Gamma) = \mathrm{rank}(C) = \mathrm{rank}(DF(w))$. Since $\ker(I(H)|_\Gamma) \subset \ker(DF(w))$, $\ker(DF(w)) = \ker(I(H)|_\Gamma)$. □

Let $x(t)$ denote the interpolated processes of $x(n)$:

$$x(t) = \sum_{n\geq 0}\left(x(n) + \frac{t-\tau_n}{\gamma_n}(x(n+1)-x(n))\right) 1|_{[\tau_n,\tau_{n+1})}(t)$$

where $\tau_n = \sum_{k=0}^{n-1}\gamma_k$. If we prove the convergence $x(t)$, then the convergence of $x(n)$ follows immediately.

The following lemma tells us how well we can approximate the interpolation process by the semiflow $\{\Phi_t\}_{t\geq 0}$ induced by F.

Lemma 5.4. *[Ben99, Proposition 8.3] Almost surely,*

$$\sup_{T>0}\limsup_{t\to\infty}\frac{1}{t}\log\left(\sup_{0\leq h\leq T} d(x(t+h),\Phi_h(x(t)))\right) \leq -1/2.$$

This lemma gives a quantitative estimation of how well the interpolated process can be approximated by the semi-flow Φ of F. In the original proposition, the right hand of the inequality is equal to $\frac{1}{2}\lim_{n\to\infty}\frac{\log(\gamma_n)}{\tau_n}$. But, as γ_n is $O(1/n)$, $\frac{1}{2}\lim_{n\to\infty}\frac{\log(\gamma_n)}{\tau_n} = -\frac{1}{2}$.

Fix a subset $I \subset int(\mathcal{J})$, and a small neighborhood U of I in Δ, there is a foliation of submanifolds $\{\mathcal{F}_x\}_{x\in U}$ such that:

- $\mathcal{F}_x \pitchfork \mathcal{J}$ is one point denoted by $\pi(x)$.
- For each x, the flow on $\mathcal{F}_x$ exponentially contracts to $\pi(x)$.

This is an application of the theory of invariant manifolds for normally hyperbolic sets (see [HPS77, Theorem 4.1]).

That submanifold induces a map $\pi : U \to \mathcal{J}$. Notice that π is not a projection (it is not even linear), but $\mathcal{F}_x$ depends smoothly on x. Hence if U is small, then π is 2-Lipschitz:

$$d(\pi(x),\pi(y)) \leq 2d(x,y),\ \forall x,y\in U. \tag{3}$$

Let fix a $\epsilon > 0$ and reduce U, if necessary, so that

$$U = \{x\in\Delta : \pi\in I \text{ and } d(x,\pi(x)) < \epsilon\}, \tag{4}$$

Let $c = \max\{\lambda : \lambda \neq \text{ is a eigenvalue of } DF(w),\ x\in I\}$. By Lemma 5.3, $c < 0$. Thus there is a $K > 0$ such that

$$d(\Phi_t(x),\pi(x)) \leq Ke^{ct}d(x,\pi(x)), \forall x\in U, \forall t\geq 0. \tag{5}$$

By hypothesis, $x(n)$ doesn't converge to $\partial\Delta$, and since $int(\mathcal{J}) \cap int(\Delta) = int(\mathcal{J})$, by Lemma 5.1, it must have an accumulation point in the $int(\mathcal{J})$. Let $I \subset int(\mathcal{J})$ be a subset that contains this point and U as in (4)

Lemma 5.5. *[CL14, Lemma 4.4] Let $x(t) \in U$. If t, T are large enough, then*

(i) $d(\pi(x(t+T)), \pi(x(t))) \leq 2e^{-\frac{t}{4}}$

(ii) $x(t+T) \in U$

Proof. To simplify, denote $x(t)$ by x and $x(t+T)$ by $x(T)$.

(i) Since $\pi(\Phi_T(x)) = \pi(x)$ and π is 2-Lipschitz:

$$d(\pi(x(T)), \pi(x)) = d(\pi(x(T)), \pi(\Phi_T(x))) \leq 2d(x(T), \Phi_T(x))$$

By Lemma 5.4, $d(x(T), \Phi_T(x)) \leq e^{-\frac{t}{4}}$ for large t, so $d(\pi(x(T)), \pi(x)) \leq 2e^{-\frac{t}{4}}$.

(ii) By triangular inequality,

$$\begin{aligned} d(x(T), \pi(x(T))) &\leq d(x(T), \Phi_T(x)) + d(\Phi_T(x), \pi(\Phi_T(x))) + \\ &\quad d(\pi(\Phi_T(x)), \pi(x(T))) \\ &\leq 3d(x(T), \Phi_T(x)) + d(\Phi_T(x), \pi(x)) \\ &\leq 3e^{-\frac{t}{4}} + Ke^{cT} d(x, \pi(x)) \\ &\leq 3e^{-\frac{t}{4}} + Ke^{cT} \epsilon \\ &< \epsilon \end{aligned}$$

whenever $3e^{-\frac{t}{4}} < \frac{\epsilon}{2}$ and $Ke^{cT} < \frac{1}{2}$. □

This second part of the lemma allows us to apply it inductively to the points $x_k := x(t+kT)$, $k \geq 0$. If $x_k \in U$, by the Lemma 5.5, then $x_{k+1} \in U$ and $d(\pi(x_{k+1}), \pi(x_k)) < 2e^{-\frac{t+kT}{4}}$. For this, we can choose t and T large enough so that $2\sum_k e^{-\frac{t+kT}{4}}$ can be arbitrary small impliying that $\pi(x_k) \in I, \forall k \geq 0$. Thus $\pi(x_k)$ converges. Let $\tilde{x}$ be the limit of $\pi(x_k)$.

Notice that in the proof of Lemma 5.5(ii), the following inequality holds for all $k \geq 0$:

$$d(x_k, \pi(x_k) \leq 3e^{-\frac{t+(k-1)T}{4}} + Ke^{cT} d(x_{k-1}, \pi(x_{k-1})) \tag{6}$$

Let $\lambda = Ke^{cT}$, thus:

$$\begin{aligned} d(x_k, \pi(x_k)) &\leq 3e^{-\frac{t}{4}} \left(e^{-\frac{(k-1)T}{4}} + \lambda e^{-\frac{(k-2)T}{4}} + \cdots + \lambda^{k-1} \right) + \lambda^k d(x_0, \pi(x_0)) \\ &\leq 3e^{-\frac{t}{4}} k \left(\max\left\{ e^{-\frac{T}{4}}, \lambda \right\} \right)^{k-1} + \lambda^k d(x_0, \pi(x_0)). \end{aligned}$$

For T large enough, $\max\left\{ e^{-\frac{T}{4}}, \lambda \right\} < 1$, hence $\lim d(x_k, \pi(x_k)) = 0$. By triangular inequality, $d(x_k, \tilde{x}) \leq d(x_k, \pi(x_k)) + d(\pi(x_k), \tilde{x})$, and taking the limit on both sides, $\lim d(x_k, \tilde{x}) = 0$. Thus $\lim x_k = \tilde{x}$.

Now let $s \in [t + kT, t + (k+1)T]$, by the triangular inequality and Lemma 5.4:

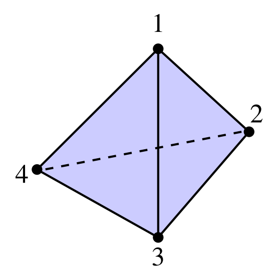
$$\begin{aligned} d(x(s), \tilde{x}) &= d(x(s), \Phi_{s-(t+kT)}(\tilde{x})) \\ &\leq d(x(s), \Phi_{s-(t+kT)}(x_k)) + d(\Phi_{s-(t+kT)}0(x_k), \Phi_{s-(t+kT)}(\tilde{x})) \\ &\leq e^{-\frac{t+kT}{4}} + c(T) d(x_k, \tilde{x}), \end{aligned}$$

where $c(T) > 0$ is the supremum of the Lipchitz constants of $\Phi_\delta, \delta \in [0, T]$. Therefore $x(t)$ converges to $\tilde{x}$, and this concludes the proof of part (b) and of the theorem.

6 Some examples

6.1 Platonic solids

We can view the platonic solids as hypergraphs where the faces of the solids are the hyperedges. For example:



In this case, we consider a tetrahedron with vertices $V = \{1, 2, 3, 4\}$ and hyperedges $E = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.
Considering the other platonic solids, we have the following table:

Hypergraph H	$\dim \ker I(H)$	$\dim \ker I(H)\|_\Gamma$
Tetrahedron	0	0
Cube	4	≥ 3
Octahedron	2	≥ 1
Icosahedron	0	0
Dodecahedron	8	≥ 7

The uniform measure is always a non-unstable equilibrium for a uniform and regular hypergraph. In particular, it is the only non-unstable equilibrium for the tetrahedron and the icosahedron.

6.2 Translated properties of Graphs to Hypergraphs

According to the Corollary 1.3 of [BBCL15], a finite, connected, regular, non-bipartite graph has a finite equilibria set, so it is natural to ask if the equilibria set of a finite, connected, regular, k-uniform, non-k-partite hypergraph is also finite

Example 1. *Consider the hypergraph with the following incidence matrix*

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

This matrix represents a 4-uniform, 4-regular, non-4-partite hypergraph. Its rank is equal to 5, therefore Λ is a 2-dimensional subset of the 6-dimensional simplex, which is an infinite equilibria set. In other words, this result can't be generalized for hypergraphs.

There are also examples of a k-uniform, regular, k-partite hypergraph, and a k-uniform, non-regular, non-k-partite hypergraph that has infinite equilibria sets.

6.3 Hypergraphs with tight-cycles

Definition. *If there are vertices $v_1, \dots, v_k$ and hyperedges $E_i = \{v_i, \dots, v_{i+l-1}\}$ (considering the indices modulo k), with $i \in [k]$, then a pair (V^k, E^k), $V^k = \{v_1, \dots, v_k\}$ and $E^k = \{E_1, \dots, E_k\}$, is a (k, l)-tight-cycle, $k \geq l$.*

Theorem 6.1. *Let H be a hypergraph with m vertices. If H contains a (m, l)-tight-cycle and $\gcd(m, l) = d$, then $\dim \ker(I(H)) \leq d - 1$.*

Proof. For this proof, we will consider the indices modulo m. From $I(H) \cdot x = 0$, the rows of $I(H)$ corresponding to the hyperedges of the tight-cycle give us $\sum_{i=j}^{j+l-1} x_i = 0$, $j \in \{1, \dots m\}$. We get $x_i = x_{i+l}$ for every $i \in [k]$. Thus, if $i \equiv j \pmod l$, then $x_i = x_j$.

If $i \equiv j \pmod d$, there exists an integer k such that $j = i + kd$. By Bezout's Theorem, there exist integers p, q satisfying $pm + ql = kd$, so $j = i + pm + ql$. Therefore, if $i \equiv j \pmod d$, then $x_j = x_{i+pm+ql} = x_{i+ql} = x_i$.

Writing $m = m'd$, we define $B_i = \{i, i + d, \dots, i + (m' - 1)d\}$, such that $[m] = \bigcup_{i=1}^{d} B_i$ is a partition. So, if $j \in B_i$, then $x_j = x_a$ for all $a \in B_i$. Thus, the equation $x_i + \dots + x_{i+l-1} = 0$ is simplified to $\frac{l}{d}(x_i + \dots + x_{i+d-1}) = 0$. In particular, $x_1 + \dots + x_d = 0$. Therefore, $\dim(\ker(I(H))) \leq d - 1$.

□

Corollary 1. *Let H be a hypergraph with m vertices. If H contains a (m, l)-tight-cycle and $\gcd(m, l) = 1$, then $I(H)$ is injective.*

Example 2. *Let G and H be hypergraphs given by a pyramid with a 3-polygonal and 4-polygonal base, respectively. The incidence matrices are*

$$
I(G) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad I(H) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.
$$

Given the systems $I(G) \cdot x = 0$ and $I(H) \cdot x = 0$, the last line concludes that $x_1 = 0$, and the case reduces to the $(3,2)$-tight-cycle and $(4,2)$-tight-cycle, respectively. We can generalize this to hypergraphs with n vertices and a $(n-1,l)$-tight-cycle.

7 Degree and boundary

Let $w \in \Lambda$ be a non-unstable equilibrium point and denote by d_i the degree of the vertex i. We may ask if there is any relation between a vertex degree and its coordinate in w. A simple and natural way to try to relate the degrees with the proportions is

$$
d_i > d_j \implies w_i \geq w_j, \forall i,j \in V.
$$

If it were true, it would be saying that the urns that compete for more balls will have a bigger proportion of balls at the limit. To approach this question, we will consider the case where we have a vertex that belongs to only one edge. In graph theory, this kind of vertex is called a leaf.

Proposition 1. *Let $H = (V, E)$ be a connected and finite hypergraph with $V = [m]$ and $|E| = N$. Assume that the vertex $i \in V$ is a leaf. If $w = (w_1, w_2, \dots, w_m)$ is an equilibrium point of F with $w_i > 0$, then w is unstable.*

Proof. Since w is an equilibrium point, we have that $F(w) = w_i \frac{\partial L}{\partial v_i}(w) = 0$. By hypothesis $w_i > 0$, so we must have $\frac{\partial L}{\partial v_i}(w) = 0$. Note that

$$
\frac{\partial L}{\partial v_i}(w) = -1 + \frac{1}{N w_{I_0}} \Rightarrow w_{I_0} = \frac{1}{N},
$$

where I_0 is the only hyperedge that contains i. Thus, for every j adjacent to i, we have

$$
\frac{\partial L}{\partial v_j}(w) = -1 + \frac{1}{N}\left(\frac{1}{w_{I_0}} + \sum_{\substack{I \in E^j \\ I \neq I_0}} \frac{1}{w_I} \right) = \frac{1}{N} \sum_{\substack{I \in E^j \\ I \neq I_0}} \frac{1}{w_I}
$$

For the domain of F, we know that $w_I > 0, \forall I \in E^j, I \neq I_0$. Then, we must have $w_j = 0$, otherwise, w wouldn't be an equilibrium point. By Lemma 7.1, we conclude that w is unstable. $\square$

Now we will give an example of a graph, which is simply a 2-uniform hypergraph, where the answer to the question is negative.

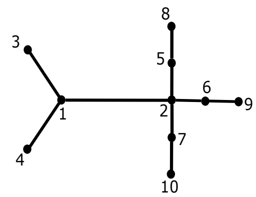


Figure 1: Graph G. There is an equilibrium point v in which $v_1 > v_2$.

Example 3. *Consider the graph G represented in figure 1. In G, we have $d_1 = 3 < 4 = d_2$ and a stable equilibrium point v where*

$$v_1 = \frac{26}{135} + \frac{2\sqrt{34}}{135} > \frac{4}{135} + \frac{11\sqrt{\frac{17}{2}}}{135} = v_2.$$

Note that vertices 5, 6, and 7, which are adjacent to 2, are also adjacent to a leaf each. By Proposition 2, the proportion of balls in the leaves has a limit equal to zero. Then, there is a moment where the vertices adjacent to these leaves start to get almost every ball from them. Because of it, we may call the leaves "weak" vertices and call the vertices 5, 6, and 7 "strong" vertices. This way, we may understand the example by noticing that 1 is adjacent to 2 and the other two "weak" vertices, while 2 is adjacent to 1 and the other three "strong" vertices.

In the previous example, we see that 2 is "weaker" than 1 and, in that case, it's happening because 2 is two vertices away from the boundary of the graph while 1 is only one vertex away from the boundary of the graph. We can also see an example where a vertex is further away from the boundary than some other vertex, but it is still "stronger" than the other one.

Example 4. *In graph G_1 of figure 2, we have a stable equilibrium point w where the leaves are "weak"($w_i = 0$, for every i that is a a leaf), the vertices 2, 3, 4 and 5 are "strong" ($w_2 = \cdots = w_5 = 1/4$) and the vertex 1 is also "weak"($w_1 = 0$). But if we look at G_2, we have a stable equilibrium point w' where the leaves are "weak"($w_i = 0$, for every i that is a leaf), the vertices $6, 7, \ldots, 17$ are "strong"($w_6 = \cdots = w_{17} = 1/13$), the vertices $2, 3, 4 and 5$ are "weak"($w_2 = \cdots = w_5 = 0$) and the vertex 1 is also "strong"($w_1 = 1/13$) even being further away from the boundary than 2, 3, 4 and 5.*

8 Completion of Incidence Matrix by Wavelets

Given β, the base of $\mathbb{R}^n$, formed by vectors of 0's and 1's such that each vector has at least two components different from zero, and an incidence matrix $I(H)$ of a hypergraph H, it is possible to insert vectors of β in $I(H)$ in such a way the rank of the matrix be equal the vertices number. In the case that rank of $I(H)$ is equal to the vertices number, it is known there is a unique deterministic point $v = v(H)$ that $x(n)$ converges to v almost surely. Otherwise, can not conclude the same results.

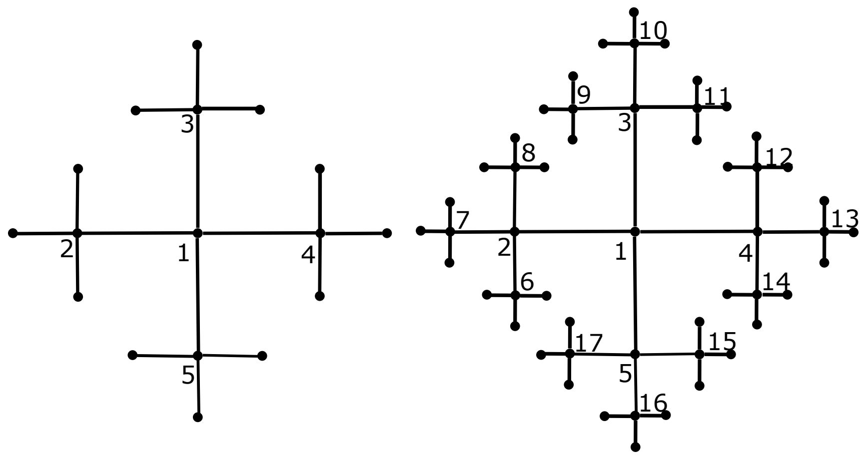


Figure 2: We have graph G_1 on the left and graph G_2 on the right.

Thus, using the Completion of Incidence Matrix Method, the objective is to modify H, in such a way the new rank of $I(H)$ is equal to the vertices number and conclude the new Pólya's Urn problem has a unique deterministic solution.

Definition. *A wavelets matrix of order n, denoted by* $\mathbb{W}(n)$*, is given by*

$$\begin{cases} a_{ij} = 1, \ \textit{if } j - 1 \equiv 0 \pmod{i} \\ a_{ij} = 0, \ \textit{otherwise} \end{cases} .$$

Proposition 2. *The vector os* $\mathbb{W}(i)$ *form a basis of* $\mathbb{R}^i$ *for* $i \in \{1, 2, 3\}$.

Proposition 3. *The vectors of* $\mathbb{W}(n)$ *form a basis of* $\mathbb{R}^n$.

Proof. It is enough to show that $\ker(\mathbb{W}(n)) = \{0\}$. Considering the system $\mathbb{W}(n) \cdot x = 0$, from the last row we get $x_1 = 0$. Now, if $i > \dfrac{n-1}{2}$, then row i of the system is $x_1 + x_{i+1} = 0$ implying $x_{i+1} = 0$. So, we can reduce $\mathbb{W}(n)$ to the case $\mathbb{W}(n')$, such that $n' = \min \left\{ i \,\middle|\, i > \dfrac{n-1}{2} \right\}$.

We can repeat the process up to $\mathbb{W}(i)$, $i \in \{1, 2, 3\}$, and by Proposition 2, the vectors of $\mathbb{W}(n)$ form a basis of $\mathbb{R}^n$. □

In consequence of previous results, we can complete the incidence matrix $I(H)$ of a hypergraph H with wavelets vectors to form a basis of $\mathbb{R}^n$. It concludes the Completion of Incidence Matrix by Wavelets Method.

9 Futher questions

It's not clear the behavior when $x(n)$ converges to $\partial\Delta$, because we can't guarantee that $\ker(I(H))|_{T_w\Delta} = \ker(DF(w))$ as in Lemma 5.3, when $w \notin \Lambda_{[m]}$. But we expected the behavior to be the same, and conjecture the following.

Conjecture. *Let H be a finite connected hypergraph. Then there is a closed subset of Δ, $\mathcal{J} = \mathcal{J}(H)$, such that $x(n)$ converges to a point of $\mathcal{J}$ almost surely.*

Besides this, a question that remains wide open is the distribution of the limit of $x(n)$. Even for graphs, no progress has been made in the direction.

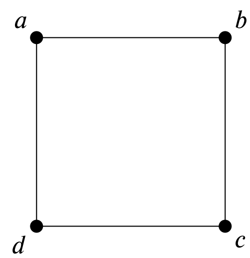
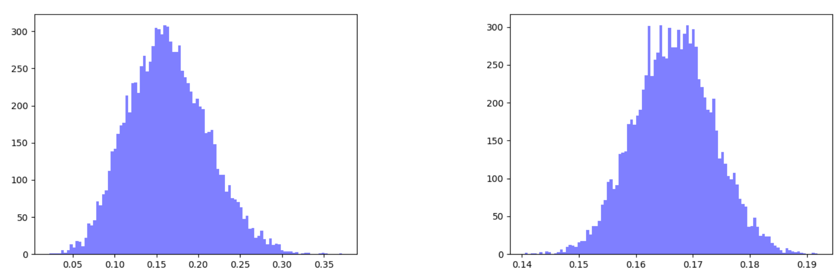


Figure 3: Graph named as square

We made some plots for the case of the square, trying to understand the behavior for the easiest case, and this lead us to the following conjecture.



(a) $B(0) = (4, 8, 4, 8)$

(b) $B(0) = (200, 400, 200, 400)$

Figure 4: Simulations of the distribution of the limit of $x(n)$ for the square

Conjecture. *Let $p(B)$ be the distribution of the square with initial balls equal to $B = (a, b, c, d)$, and $p^*(B)$ be the mode of $p(B)$. Fix $k = \frac{b}{a}$ and the initial balls equals to $\bar{B} = (a, ka, a, ka)$, then $\lim_{a\to\infty} p^*(\bar{B}) = \dfrac{1}{2+2k}$.*

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