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Intransitive Dice: A Report on the Developments at Jornadas de Pesquisa

São Carlos, São Paulo

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1 Introduction

Imagine three different dices A, B and C. Two players start a game where each one choose a dice and roll it, the one who gets the higher value on the dice wins the game, and if that is a tie, they must roll the dice again until there is a winner. The intransitive question is: is it possible to create dice A, B an C such that A is better than B, B is better than C and C is better than A? The answer is, surprisingly, yes! A set of dice with such property is called non-transitive or intransitive set of dice, or intransitive dice, for short.

Recently, some research has been developed on the study of intransitive dice, including a project by Polymath[1]. In this study, among other things, the group examined the probability of a die A being better than a die C, given that A is better than die B and B is better than die C.

We characterize the existence of intransitive sets of fair dice in general and in a new, more natural model of dice called the *weighted model*, in which a biased die with n sides has faces with numbers from 1 to n.

The main result of this text is a version of the celebrated central limit theorem. Let $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, and $(C_n)_{n\in\mathbb{N}}$ be sequences of iid random variables with a uniform distribution on the interval $(0,1)$, and let X, Y, and Z be standard normal variables such that

$$\operatorname{Cov}(X,Y) = \operatorname{Cov}(X,Z) = \operatorname{Cov}(Y,Z) = -\frac{1}{2}.$$

If we define

$$\begin{aligned}
\mathcal{A}_n &= \sum_{i=1}^{n}\sum_{j=1}^{n} \chi_{A_i>B_j},\\
\mathcal{B}_n &= \sum_{i=1}^{n}\sum_{j=1}^{n} \chi_{B_i>C_j},\\
\mathcal{C}_n &= \sum_{i=1}^{n}\sum_{j=1}^{n} \chi_{C_i>A_j}
\end{aligned}$$

and if $\widetilde{\mathcal{A}}_n$, $\widetilde{\mathcal{B}}_n$, and $\widetilde{\mathcal{C}}_n$ are their respective standardized versions, the following theorem holds.

Theorem 5.0.3. *The random vector $(\widetilde{\mathcal{A}}_n, \widetilde{\mathcal{B}}_n, \widetilde{\mathcal{C}}_n)$ converges in distribution to (X,Y,Z) as $n \to \infty$.*

A direct consequence of this theorem is that the probability of three dice being intransitive tends to 0 as n grows

2 Definitions

An n-sided die is a pair (A, α), where A is the vector $(A_1, \ldots, A_n)$, with each A_k being the number on the face k, and α is a random variable taking values on $\{1, 2, ..., n\}$. The die is said to roll the face k with probability $\mathbb{P}(\alpha = k)$ and results A_k. If this probability equals $1/n$ for every k, the die is said to be honest, fair, or unbiased. Otherwise, it is said to be unfair or biased (the reason for the latter will be clarified afterward). If there is no ambiguity, the die will be denoted as A, and in that case, it is useful to denote the random result of A in a roll by $\rho(A)$.

A die A is said to be **better than** a die B, and it is denoted by $A \rhd B$ if the probability of A rolling a higher value than B is greater than the probability of B rolling a higher value than A. To the same extent, the die B is said to be **worse than** A, and it is denoted by $B \lhd A$.

An indexed family $\{D^{(1)}, D^{(2)}, \ldots, D^{(n)}\}$ of dice is said to be intransitive if there exists a permutation σ such that $D^{\sigma(1)} \rhd D^{\sigma(2)} \rhd \ldots \rhd D^{\sigma(n)} \rhd D^{\sigma(1)}$. Note that while $\rhd$ is an asymmetric relation, it is not necessarily transitive, so it does not define an order relation.

Although it is usually noted the similarity between specifics sets of intransitive dice and the game "Rock, Paper, Scissors" and its variations, such as "Rock, Paper, Scissors, Lizard, Spock" (RPSLS, for short. See, for example, [2] and [3]), there is an intrinsic difference between the two games, in a sense that, while paper always beats rock, rock always beats scissors and so on, there still exists the possibility of a die A being beaten by a die B, even though A is better than B. Figure 1 exhibits diagrams of professor James Grime's set of intransitive dice [2] and the game of RPSLS and the respective probability of "items" beating each other.

It then raises a question, is it possible to construct a set of n dice $D^{(1)}$, $D^{(2)}$, $\ldots$, $D^{(n)}$ such that it closely resembles a game of paper, scissors, or rock? That is, is it possible that $D^{(1)}$ almost always beats $D^{(2)}$, $D^{(2)}$ almost always beats $D^{(3)}$, in general, the die $D^{(k)}$ almost always beats the die $D^{(k+1)}$, and $D^{(n)}$ almost always beats $D^{(1)}$? That is not possible.

Theorem 2.0.1. *Let $D^{(1)}, \ldots, D^{(n)}$ be a set of dice such that $\mathbb{P}(\rho(D^{(k)}) > \rho(D^{(k+1)})) > 1-\varepsilon$, for each $1 \leq k \leq n-1$ and some $\varepsilon > 0$. Then, $\mathbb{P}(\rho(D^{(n)}) > \rho(D^{(1)})) < (n-1)\varepsilon$.*

Proof. Let E_1, E_2, and E_3 be the events in which $\rho(D^{(1)}) > \rho(D^{(2)})$, $\rho(D^{(2)}) > \rho(D^{(3)})$ and $\rho(D^{(1)}) > \rho(D^{(3)})$ respectively.

It is evident that

$$\mathbb{P}(E_1 \cap E_2) \leq \mathbb{P}(E_3)$$

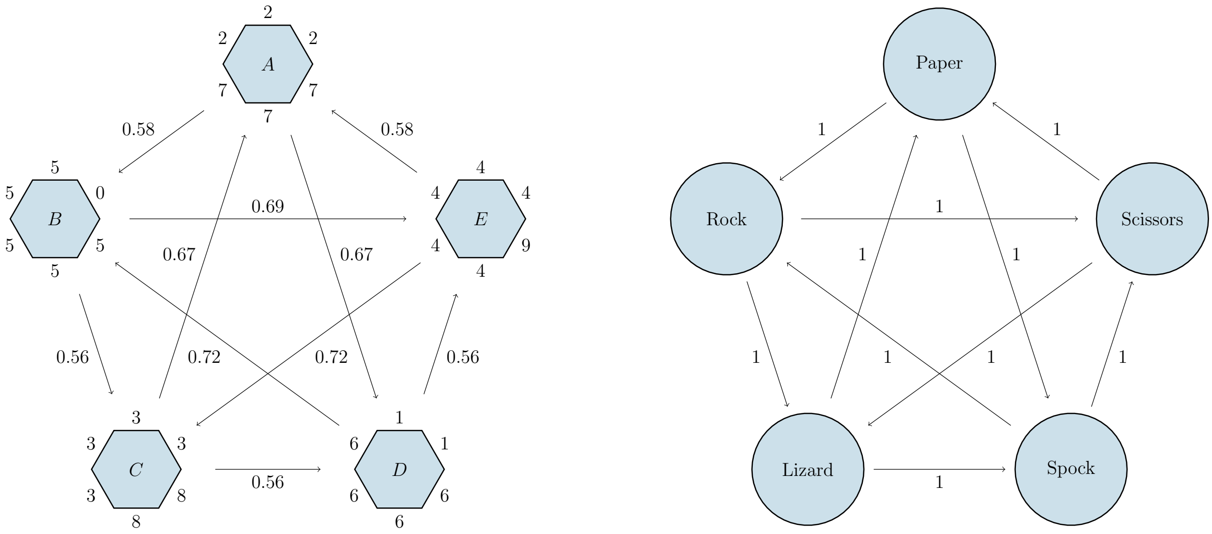


Figure 1 – Grime's dice (left) and the game of Rock, Paper, Scissors, Lizard, Spock (right)

and, by the inclusion-exclusion principle,

$$\begin{aligned}\mathbb{P}(E_1 \cap E_2) &= \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cup E_2) \\ &> 1 - 2\varepsilon\end{aligned}$$

Applying the same argument recursively for $D^{(1)}, D^{(k)}, D^{(k+1)}$,

$$\mathbb{P}(\rho(D^{(1)}) > \rho(D^{(n)})) > 1 - (n-1)\varepsilon,$$

and the result is immediate. $\square$

3 Deterministic Models of Dice

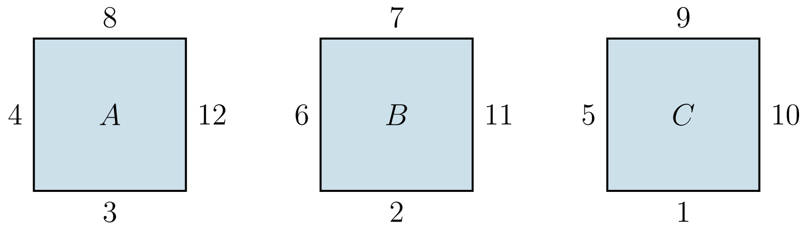
3.1 Dice as Strings

The case where the dice are honest, no faces are shared between two dice and no dice have repeated faces is of particular interest since it enables the use of different representations for a set of dice. One of which is denominated the string representation. [1]

Suppose that $D^{(1)}, D^{(2)}, \ldots, D^{(m)}$ are honest n-sided dice such that there are no repeating values on the faces through all dice. Therefore, it is possible to sort all the faces of the dice by value inputs and create a descending sequence that is given by some permutation of the faces of $D^{(1)}, D^{(2)}, \ldots, D^{(m)}$. It then generates a unique string that represents the set of

[1] The idea of translating the dice as strings was inspired by the excellent video made by the YouTube channel Polylog: "We designed special dice using math, but there's a catch".

dice. This process for three 4-sided dice is represented in Figure 2. In this chapter, all the dice will always be treated as honest.



A	12				8				4	3		
B		11				7	6				2	
C			10	9				5				1
	a	b	c	c	a	b	b	c	a	a	b	c

Figure 2 – An example of a triple of dice and its representation as a string.

Note that this sequence makes explicit that, when comparing the faces of the dice, one only cares about the relative position of the value in this sequence created. Therefore it is possible to exchange values by some symbolic representation of the dice the value belongs to without losing the information necessary to compare the dice. With the string, to compare die $D^{(2)}$ to $D^{(1)}$ is to sum how many letters $s^{(1)}$ are to the right of every letter $s^{(2)}$. The result is how many possible victories $D^{(2)}$ has over $D^{(1)}$. This relation $D^{(1)} \triangleright D^{(2)}$ has a natural translation when comparing strings; therefore, the same notation may also be used in this context.

In the depiction of Figure 2, the obtained string uses a, b, and c to represent values of the respective A, B, and C die. This gives *abccabbcaabc*. It is possible to extend this process to any number of dice with any number of faces, given that they fulfill the requirement of not repeating values between them or in themselves.

To compare only two dice of the string, it is enough to remove all letters that are not representative of the dice of interest, and without loss, one can compare the two within the sub-sequence created. In the example given, to compare the dice A and B, analyze the sub-sequence generated by removing the c's: *abccabbcaabc* $\rightarrow$ *ababbaab*. Again, the problem is resolved by comparing the relative position of the letter in the string. Sum how many b's are to the right of every a to obtain the number of victories of a over b and vice-versa.

Definition 3.1.1. *A string is said to be of characteristic m and denoted as S_m if it is* *composite of exactly m distinct letters.*

In the particular case where all m distinct letters on a string have the same number n of occurrences, this string is said to be balanced and of order n, and shall be denoted as $S_{m,n}$.

Particularly useful classes of strings also include the ones that correspond to sets of intransitive and neutral dice. The last correspond to sets of dice where every die beats any other die the same amount of times. Results regarding these strings include the following lemmas

Lemma 3.1.2. *Let $N_{s^{(1)}>s^{(2)}}$ be the number of wins of $s^{(1)}$ over $s^{(2)}$ in a string $S_{m,n}$, then the number of wins of $s^{(2)}$ over $s^{(1)}$ in S^* is $N^*_{s^{(2)}>s^{(1)}} = N_{s^{(1)}>s^{(2)}}$, where S^* denotes the reverse of S.*

Proof. Note that, for every sub-sequence $s^{(2)}s^{(1)}$ of $S_{m,n}$ formed by the ith $s^{(1)}$ and jth $s^{(2)}$, there is a corresponding sub-sequence $s^{(1)}s^{(2)}$ in $S^*_{m,n}$, formed by the $(m-i)$th $s^{(1)}$ and $(m-i)$th $s^{(2)}$. Since $N_{s^{(1)}>s^{(2)}}$ is the sum, for every $s^{(1)}$ on the string, of the amount of $s^{(2)}$'s on $S_{m,n}$, it follows immediately that $N^*_{s^{(2)}>s^{(1)}} = N_{s^{(1)}>s^{(2)}}$. □

Lemma 3.1.3. *If $S_{m,2l}$ is a symmetric string where every letter $s^{(i)}$ has exactly $2l$ occurrences, then $S_{m,2l}$ is neutral.*

Proof. Since $S_{m,2l}$ is symmetric, it can be written as the concatenation of a string $S_{m,l}$ and it's reverse $S^*_{m,l}$.

It is valid to write for each pair of letters $s^{(i)}$ and $s^{(j)}$ that

$$N_{s^{(i)}>s^{(j)}} + N^*_{s^{(i)}>s^{(j)}} + m^2 = N_{s^{(j)}>s^{(i)}} + N^*_{s^{(j)}>s^{(i)}} + m^2$$

Where terms on the expression came from $S_{m,l}$ and $S^*_{m,l}$ and a coupling term.

By Lemma 3.1.2 it is known that $N_{s^{(i)}>s^{(j)}} = N^*_{s^{(j)}>s^{(i)}}$ and that $N_{s^{(j)}>s^{(i)}} = N^*_{s^{(i)}>s^{(j)}}$, proving the Lemma. □

Lemma 3.1.4. *Let $S_{m,n}$ be an intransitive string. Then there is a string $S_{m+1,n}$ that is also intransitive.*

Proof. Suppose, without loss of generality, that $s^{(1)} \triangleright \ldots \triangleright s^{(m)} \triangleright s^{(1)}$, and let $S_{(m+1),n}$ be the string obtained by replacing every occurrence of $s^{(m)}$ by $s^{(m)}s^{(m+1)}$.

Note that the relation between any two letters in $s^{(1)}, \ldots, s^{(m)}$ is preserved in $S_{(m+1),n}$, also, note $s^{(m+1)} \triangleright s^{(1)}$ in $S_{(m+1),n}$, as the sub-sequence consisting only of $s^{(m+1)}$ and $s^{(1)}$, is equivalent to the one formed only by $s^{(m)}$ and $s^{(1)}$.

It is easy to see that $s^m \rhd s^{m+1}$, as the sub-sequence formed by them, is of the form $\underbrace{s^m s^{m+1} \dots s^m s^{m+1}}_{n}$, and the number of s^{m+1} at the left of each s^m is always bigger than the converse.

Therefore, $S_{(m+1),n}$ is intransitive with $s^{(1)} \rhd \dots \rhd s^{(m)} \rhd s^{(m+1)} \rhd s^{(1)}$. $\square$

Notice that by inserting a new die, the values of the faces of each die are possibly shifted, so there's no guarantee that any die that was part of the first intransitive set also belongs to the second one.

Lemma 3.1.5. *A string $S_{3,2}$ can not be intransitive.*

Proof. Let $S_{3,2}$ be a string in which $s^{(1)} \rhd s^{(2)} \rhd s^{(3)}$. This way, the letters $s^{(2)}$ must come at least three times at the right of the letters $s^{(1)}$. That implies that at least one letter $s^{(1)}$ has to come at the left of both letters $s^{(2)}$. In the same way, at least one letter $s^{(2)}$ has to come at the left of both letters $s^{(3)}$. But if that happens, one can conclude that at least one letter $s^{(1)}$ comes at the left of both letters $s^{(3)}$, and this set can not be intransitive.

$\square$

Lemma 3.1.6. *Let $S_{m,n}$ and $I_{m,2k}$ common lettered strings be intransitive and neutral, respectively. Then $I_{m,2k}S_{m,n}$ is an intransitive string.*

Proof. Similarly to the demonstration of 3.1.3, the juxtaposition of $I_{m,2k}$ introduces $\frac{(2k)^2}{2}$ victories of any letter $s^{(1)}$ on $I_{m,2k}$ over any other letter $s^{(2)}$ in $I_{m,2k}$, and $2kN_{s^{(1)}>s^{(2)}}$ victories of $s^{(1)}$'s on I_{2k} over $s^{(2)}$'s on S_n. So, if

$$N_{s^{(1)}>s^{(2)}} > N_{s^{(1)}>s^{(2)}}$$

The following result holds

$$\underbrace{2k^2 + 2kN_{s^{(1)}>s^{(2)}} + N_{s^{(1)}>s^{(2)}}}_{N'_{s^{(1)}>s^{(2)}}} > \underbrace{2k^2 + 2kN_{s^{(1)}>s^{(2)}} + N_{s^{(1)}>s^{(2)}}}_{N'_{s^{(1)}>s^{(2)}}}$$

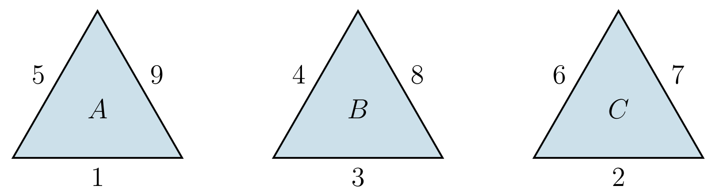
Since the concatenation preserved the inequalities, it follows that $I_{2k}S_n$ is intransitive. $\square$

Lemma 3.1.7. *Let $S_{m,n}$ be an intransitive string. Then there is $S_{m,(n+2)}$ that is also intransitive.*

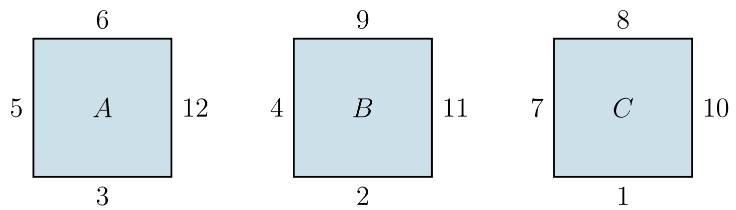
Proof. Let S_n be an intransitive string. In the interest of creating a new string of higher order, take the juxtaposition of S_n and a symmetric string as done in Lemma 3.1.6. By choosing the lowest order of such a string, the new intransitive string will represent a set of $(n+2)$-sided dice. $\square$

Theorem 3.1.8. *For every $n \geq 3$ and $m \geq 3$ there are intransitive strings of characteristic m and order n.*

Proof. By Lemma 3.1.5, there is no intransitive string that represents three 2-sided dice. It is sufficient to show that exists an intransitive string representing three 3-sided dice, such as the first example in Figure 3 and an intransitive string representing three 4-sided dice, such as the second example in Figure 3. The rest follows from induction on m and n from both examples.



abccabbca



abcbccaababc

Figure 3 – Set of three three-sided and four-sided intransitive dice

□

Table 1 shows the existence proved in theorem 3.1.8. It is, as shown, possible to determine all possible cases of existence created by combining m and n.

3.2 Weighted Model

The most natural way to create a die is to consider an n-sided die with numbers from 1 to n on its faces. At first, it does not seem possible to construct a set of intransitive dice since all dice would be the same. But it is possible if the dice are biased.

An n-sided biased die (A, α) is non-degenerate if $\mathbb{P}(\alpha = k) > 0$, for every $1 \leq k \leq n$. If, furthermore, $A = (1, 2 \ldots, n)$, the die A will be simply called a weighted die. Note

$m \backslash n$	2	3	4	5	6	7	...
3	×	✓	✓	✓	✓	✓	...
4	×	✓	✓	✓	✓	✓	...
5	×	✓	✓	✓	✓	✓	...
6	×	✓	✓	✓	✓	✓	...
7	×	✓	✓	✓	✓	✓	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 1 – Existence of fair dice with no repetition

that $\mathbb{P}(\rho(A) = k) = \mathbb{P}(\alpha = k)$. In that case, for any $\lambda > 0$ the vector $\lambda \cdot (\mathbb{P}(\rho(A) = 1), \ldots, \mathbb{P}(\rho(A) = n))$ is called a weight vector of A. Conversely, if $a = (a_1, \ldots, a_n)$, with $a_i \in \mathbb{R}_+^*$, there exists only one weighted n-sided dice (A, α) such that

$$\mathbb{P}(\rho(A) = k) = a_k / S_A,$$

where $S_A = \sum_{i=1}^n a_i$, so that a is a weight vector of the die A.

If A and B are biased n-sided dice with weights $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, respectively, it is easy to show that

$$\mathbb{P}(\rho(A) > \rho(B)) = \frac{1}{S_A S_B} \sum_{i=2}^{n} \sum_{j=1}^{i-1} a_i b_j.$$

It is possible to create intransitive weighted dice, but they cannot have less than four sides.

Lemma 3.2.1. *If A, B and C are weighted 3-sided dice such that $A \triangleright B \triangleright C$, then $A \triangleright C$.*

Proof. As $A \triangleright B$,

$$\begin{aligned} &\mathbb{P}(\rho(A) > \rho(B)) > \mathbb{P}(\rho(B) > \rho(A)) \\ \implies\ & a_2 b_1 + a_3(b_1 + b_2) > b_2 a_1 + b_3(a_1 + a_2) \\ \implies\ & (a_2 + a_3)(b_1 + b_2) > (b_2 + b_3)(a_1 + a_2). \end{aligned}$$

Similarly, $(b_2 + b_3)(c_1 + c_2) > (c_2 + c_3)(b_1 + b_2)$.

Multiplying both inequalities and using the fact that $(b_1 + b_2), (b_2 + b_3) > 0$, it follows that $(a_2 + a_3)(c_1 + c_2) > (c_2 + c_3)(a_1 + b_2)$ and, therefore, $A \triangleright C$. □

It is now easy to prove that if $D^{(1)}, \ldots, D^{(n)}$ are weighted 3-sided dice, then $D^{(1)} \triangleright \ldots \triangleright D^{(n)}$ implies that $D^{(1)} \triangleright D^{(n)}$.

The next two lemmas will be useful for what follows.

Lemma 3.2.2. *If A and B are weighted n-sided dice with weight vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, such that $A \triangleright B$, then the weighted die C with weight vector $s\boldsymbol{a} + t\boldsymbol{b}$, for $s, t > 0$, satisfies $A \triangleright C \triangleright B$.*

Proof. On the one hand,

$$\begin{aligned}\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i(sa_j + tb_j) &= s\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i a_j + t\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i b_j \\ &> s\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i a_j + t\sum_{i=2}^{n}\sum_{j=1}^{i-1} b_i a_j \\ &= \sum_{i=2}^{n}\sum_{j=1}^{i-1} (sa_i + tb_i)a_j.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i=2}^{n}\sum_{j=1}^{i-1} (sa_i + tb_i)b_j &= s\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_i b_j + t\sum_{i=2}^{n}\sum_{j=1}^{i-1} b_i b_j \\ &> s\sum_{i=2}^{n}\sum_{j=1}^{i-1} b_i a_j + t\sum_{i=2}^{n}\sum_{j=1}^{i-1} b_i b_j \\ &= \sum_{i=2}^{n}\sum_{j=1}^{i-1} b_j(sa_i + tb_i).\end{aligned}$$

□

Lemma 3.2.3. *If there exist m weighted n-sided intransitive dice, then m weighted $(n+1)$-sided intransitive dice exist.*

Proof. Let $D_1 \ldots, D_m$ be the intransitive dice and $\boldsymbol{d}^{(1)}, \ldots, \boldsymbol{d}^{(m)}$ be their respective weight vectors, where $\boldsymbol{d}^{(i)} = (d_1^{(i)}, \ldots, d_n^{(i)})$ and $D_1 \triangleright \ldots \triangleright D_m \triangleright D_1$.

For each $1 \leq k \leq m$, define the function $f_k : \mathbb{R}^{m(n+1)} \to \mathbb{R}$ by

$$f_k(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}) = \sum_{i=2}^{n+1}\sum_{j=1}^{i-1}(x_i^{(k)}x_j^{(k+1)} - x_i^{(k+1)}x_j^{(k)}),$$

where $\boldsymbol{x}^{(k)} = (x_1^{(k)}, \ldots, x_{n+1}^{(k)}) \in \mathbb{R}^{n+1}$ and $\boldsymbol{x}^{(n+1)}$ is defined as $\boldsymbol{x}^{(1)}$.

Now, if $\boldsymbol{d} = (\boldsymbol{d}^{(1)}, 0, \boldsymbol{d}^{(2)}, 0, \ldots, \boldsymbol{d}^{(m)}, 0)$, the intransitivity of the dice implies that $f_k(d) > 0$, for every k.

As, for each k, f_k is a continuous function, there is an $\varepsilon_k > 0$ such that if $\|\boldsymbol{x} - \boldsymbol{d}\|_\infty < \varepsilon_k$ then $f_k(\boldsymbol{x}) > 0$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_m\}$ and $\boldsymbol{d}' = (\boldsymbol{d}^{(1)}, \varepsilon/2, \boldsymbol{d}^{(2)}, \varepsilon/2, \ldots, \boldsymbol{d}^{(m)}, \varepsilon/2)$. Then $f_k(d') > 0$ for each k and if D_i' is defined as a biased $(n+1)$-sided die with weight vector $(\boldsymbol{d}^{(i)}, \varepsilon/2)$, the dice $D_1', \ldots, D_m'$ satisfy $D_1' \triangleright \ldots \triangleright D_m' \triangleright D_1'$. □

One might be interested then in determining which combinations of the number of dice and faces enable the creation of a set of intransitive dice. The Theorem 3.2.4 gives the conclusion.

Theorem 3.2.4. *For every $m \geq 3$ and $n \geq 4$, there exist m weighted n-sided dice that are intransitive.*

Proof. It suffices to show one example for $m = 3$ and $n = 4$. The rest follows by induction on m and n. Figure 4 shows such an example.

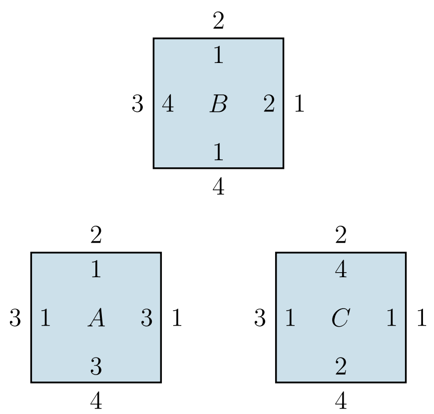


Figure 4 – A set of intransitive 4-sided biased dice. The numbers outside are the faces of the dice, and the numbers inside are the respective weights

□

It is possible to illustrate the existence of the sets in table 2. This theorem shows the existence of all possible intransitive sets created by combining m and n.

$m \backslash n$	2	3	4	5	6	7	…
3	×	×	×	×	×	×	…
4	×	✓	✓	✓	✓	✓	…
5	×	✓	✓	✓	✓	✓	…
6	×	✓	✓	✓	✓	✓	…
7	×	✓	✓	✓	✓	✓	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 2 – Existence of intransitive set of weighted dice

4 Random Models of Dice

To study different distribution properties of dice, such as the proportion of intransitive ones in a given model, it is helpful to define the concept of random dice.

Definition 4.0.1. *The n-sided die A is called a random die if its n faces A_1, A_2, …, A_n are iid random variables.*

For what follows, it is assumed that all random dice are fair.

If A and B are random n-sided dice, the number of wins of A over B given by

$$N_{A>B} = \sum_{i=1}^{n}\sum_{j=1}^{n} \chi_{A_i>B_j}$$

is now a random variable.

4.1 Continuous Uniform Distribution

Suppose $A^{(n)}$, $B^{(n)}$ and $C^{(n)}$ independent random dice with n sides in which $A_k^{(n)}$, $B_k^{(n)}$, $C_k^{(n)}$ $\sim \mathcal{U}(0,1)$, where $\mathcal{U}(0,1)^k$ denotes the uniform distribution on the set $(0,1)^k$, so that

$$(A^{(n)}, B^{(n)}, C^{(n)}) \sim \mathcal{U}(0,1)^{3n}.$$

Let $\mathfrak{D}_n$ be the subset of points of $(0,1)^{3n}$ in which the coordinates are pairwise distinct. Then, $\mathfrak{D}_n$ has measure 1 and $(A^{(n)}, B^{(n)}, C^{(n)}) \in \mathfrak{D}_n$ with total probability.

Note that if $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathfrak{D}_n$, where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in (0,1)^n$, then, using a construction similar to the one in Subsection 3.1, there exists only one string s associated to this point with n occurrences of each of the letters a, b and c, in which the letter a is associated to $\boldsymbol{a}$, the letter b is associated to $\boldsymbol{b}$ and the letter c is associated to $\boldsymbol{c}$, furthermore, the fair n-sided dice $(\boldsymbol{a}, \alpha), (\boldsymbol{b}, \beta)$ and $(\boldsymbol{c}, \gamma)$ are intransitive if, and only if, the string s is intransitive.

Define the relation R on $\mathfrak{D}_n$ such that xRy if and only if $s_x = s_y$, where s_x is the string associated to x and s_y the string associated to y. Then R is clearly an equivalence relation, and the quotient set $\mathfrak{D}_n/R$ has a natural one-to-one correspondence with the set $\mathfrak{S}_n$ of all the strings with n occurrences of each of the letters a, b and c. It is easy to check that each class in $\mathfrak{D}_n/R$ is an open set of $\mathbb{R}^{3n}$, and, therefore, is measurable.

Symmetry arguments show that each class has the same measure, so if $\mathcal{D}(n)$ is the number of equivalence classes of $\mathfrak{D}_n/R$ or, similarly, the number of strings in $\mathfrak{S}_n$, and x is any equivalence class, then

$$\sum_{y\in\mathfrak{D}_n/R} \lambda^{3n}(y) = \lambda^{3n}\left(\bigcup_{y\subset\mathfrak{D}_n/R} y\right)$$

$$
\begin{aligned}
&= \lambda^{3n}(\mathfrak{D}_n) \\
&= 1 \\
\implies \mathcal{D}(n)\lambda^{3n}(x) &= 1,
\end{aligned}
$$

where λ^k is the Lebesgue measure of $\mathbb{R}^k$. But then

$$\mathbb{P}((A^{(n)}, B^{(n)}, C^{(n)}) \in x) = \frac{1}{\mathcal{D}\,(n)}.$$

Particularly, if $\mathcal{I}\,(n)$ is the number of classes in $\mathfrak{D}/R$ represented by intransitive strings, or, analogously, the number of strings in $\mathfrak{I}_n$, where $\mathfrak{I}_n \subset \mathfrak{S}_n$ is the set of intransitive strings, and $\mathfrak{J}_n$ is the union of all these classes, then the probability of the dice $A^{(n)}, B^{(n)}$ e $C^{(n)}$ be intransitive is given by

$$\mathbb{P}((A^{(n)}, B^{(n)}, C^{(n)}) \in \mathfrak{J}_n) = \frac{\mathcal{I}\,(n)}{\mathcal{D}\,(n)},$$

so that to understand the probability of choosing intransitive dice randomly, it is enough to study the sets $\mathfrak{D}_n$ and $\mathfrak{I}_n$. The table 4.1 shows the values of $\mathcal{I}(n)$ for n between 1 and 10, calculated with the assistance of a computer.

n	$\mathcal{I}\,(n)$	$\mathcal{D}\,(n)$	$\mathcal{I}(n)/\mathcal{D}(n)$
1	0	6	0
2	0	90	0
3	30	1680	0.017857
4	78	34650	0.002251
5	10392	756756	0.013732
6	64230	17153136	0.003744
7	4186398	399072960	0.010490
8	39236706	9465511770	0.004145
9	1920331578	227873431500	0.008427
10	22545898302	5550996791340	0.004061

Table 3 – Values of $\mathcal{I}(n)$, $\mathcal{D}(n)$ and $\mathcal{I}(n)/\mathcal{D}(n)$ for $1 \leq n \leq 10$. Note how the ratios $\mathcal{I}(n)/\mathcal{D}(n)$ seem to be lower for even n. The decrease in ratios for even n may be due to neutral strings only being possible for even n, leading to fewer intransitive strings proportionally.

A simple combinatorial argument shows that

$$\mathcal{D}\,(n) = \frac{(3n)!}{(n!)^3}.$$

Additionally, using the Stirling approximation, it can be shown that

$$\mathcal{D}\,(n) \sim \frac{\sqrt{3}}{2\pi n} 3^{3n}.$$

Determining an asymptotic expression for $\mathcal{I}\,(n)$ is not so straightforward.

Note that, if $s \in \mathfrak{I}_n$ is a string such that $b \rhd a \rhd c \rhd b$, then the string s', obtained by switching all the letters a for b and vice-versa, satisfies $a \rhd b \rhd c \rhd a$, so that the set $\mathfrak{I}_n$ can be partitioned in two sets, $\mathfrak{I}_n = \mathfrak{I}_n^a \cup \mathfrak{I}_n^b$, where $\mathfrak{I}_n^a$ is the subset of strings in which $a \rhd b \rhd c$ and $\mathfrak{I}_n^b$ is the subset of strings in which $b \rhd a \rhd c$, and, therefore, there exists an one-to-one correspondence between $\mathfrak{I}_n^a$ and $\mathfrak{I}_n^b$. Then,

$$\#\mathfrak{I}_n^a = \#\mathfrak{I}_n^b = \frac{\mathcal{I}(n)}{2} =: \mathcal{I}'(n).$$

Proposition 4.1.1. *If* $r \in \mathfrak{I}_m^a$ *and* $s \in \mathfrak{I}_n^a$, *then* $rs \in \mathfrak{I}_{m+n}^a$

Proof. The new count of victories of a over b is made similarly to the one in the proof of Lemma 3.1.6, that is, if $N_{a>b}^r$, $N_{a>b}^s$ and $N_{a>b}^{rs}$ are the number of a over b victories in the strings r, s and rs, respectively, and defining $N_{b>a}^r, N_{b>a}^s$ and $N_{b>a}^{rs}$ in an analogous way, then

$$\begin{aligned} N_{a>b}^{rs} &= N_{a>b}^r + N_{a>b}^s + mn \\ &> N_{b>a}^r + N_{b>a}^s + mn \\ &= N_{b>a}^{rs} \end{aligned}$$

$\square$

Corolary 4.1.2. *If* m *and* n *are positive integers, then* $\mathcal{I}'(m+n) \geq \mathcal{I}'(m)\mathcal{I}'(n)$

Proof. If $r_1, r_2 \in \mathfrak{I}_m^a$ e $s_1, s_2 \in \mathfrak{I}_n^a$, Then it is easy to see that $r_1 s_1 = r_2 s_2$ if, and only if, $r_1 = r_2$ and $s_1 = s_2$. Thus, by the multiplication principle, $\#\{rs : r \in \mathfrak{I}_m^a \text{ and } s \in \mathfrak{I}_n^a\} = \mathcal{I}'(m)\mathcal{I}'(n)$ and the result follows from the fact that $\{rs : r \in \mathfrak{I}_m^a \text{ and } s \in \mathfrak{I}_n^a\} \subset \mathfrak{I}_{m+n}^a$. $\square$

Theorem 4.1.3. *There exists a constant* $L \in (2.314, 3\log 3]$ *such that*

$$\mathcal{I}'(n) = \mathrm{e}^{nL(1+o(1))}.$$

Proof. Suppose $m, n \geq 3$. Then

$$\log \mathcal{I}'(m+n) \geq \log \mathcal{I}'(m) + \log \mathcal{I}'(n).$$

By Fekete's lemma, there exists L satisfying

$$\lim_{n\to\infty} \frac{\log \mathcal{I}'(n)}{n} = \sup_n \frac{\log \mathcal{I}'(n)}{n} = L.$$

Equivalently,

$$\frac{\log \mathcal{I}'(n)}{n} = L(1 + o(1))$$

$$\implies \mathcal{I}'(n) = \mathrm{e}^{nL(1+o(1))}.$$

Examining the Table 4.1, it is possible to see that

$$L \geq \frac{\log \mathcal{I}'(10)}{10} \approx 2.3146.$$

On the other hand, from the fact that $\mathcal{I}'(n) \leq \mathcal{D}(n)$, it follows that

$$L = \lim_{n\to\infty} \frac{\log \mathcal{I}'(n)}{n} \leq \lim_{n\to\infty} \frac{\log \mathcal{D}(n)}{n} = 3\log 3.$$

□

The value of L is unknown, but computational simulations of $\mathcal{I}(n)/\mathcal{D}(n)$ suggest that $L = 3\log 3$.

Conjecture 4.1.4. *L is equal to* $3\log 3$.

Turning back the attention to the random variables, define $\mathcal{A}_n = N_{A^{(n)}>B^{(n)}}$, $\mathcal{B}_n = N_{B^{(n)}>C^{(n)}}$ and $\mathcal{C}_n = N_{C^{(n)}>A^{(n)}}$. See that the dice $A^{(n)}, B^{(n)}$ and $C^{(n)}$ are intransitive if, and only if, either $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n > n^2/2$ or $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n < n^2/2$, so that

$$\begin{aligned} \frac{\mathcal{I}(n)}{\mathcal{D}(n)} &= \mathbb{P}((A^{(n)}, B^{(n)}, C^{(n)}) \in \mathfrak{I}_n) \\ &= \mathbb{P}(\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n > n^2/2) + \mathbb{P}(\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n < n^2/2) \\ &= \mathbb{P}(\widetilde{\mathcal{A}}_n, \widetilde{\mathcal{B}}_n, \widetilde{\mathcal{C}}_n > 0) + \mathbb{P}(\widetilde{\mathcal{A}}_n, \widetilde{\mathcal{B}}_n, \widetilde{\mathcal{C}}_n < 0), \end{aligned} \tag{1}$$

where

$$\begin{aligned} \widetilde{\mathcal{A}}_n &= \frac{\mathcal{A}_n - n^2/2}{\sqrt{n^2(2n+1)/12}}, \\ \widetilde{\mathcal{B}}_n &= \frac{\mathcal{B}_n - n^2/2}{\sqrt{n^2(2n+1)/12}}, \\ \widetilde{\mathcal{C}}_n &= \frac{\mathcal{C}_n - n^2/2}{\sqrt{n^2(2n+1)/12}}. \end{aligned}$$

5 Central Limit Theorem for Continuous Uniform Distribution

The random variables $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$ have a close resemblance to the variables in the standard version of the Central Limit Theorem. However, the indicator functions $\chi_{A_i^n > B_j^n}$ are not an independent random variable. Thus, it is natural to search for a version of this celebrated theorem for these variables.

To state the theorem, first, it is necessary to compute the mean and the variance of the variables, which can be calculated as follows.

Note that, as $A_i^{(n)}$ and $B_j^{(n)}$ are iid and have a continuous distribution, then

$$\mathbb{P}(A_i^{(n)} > B_j^{(n)}) = \mathbb{P}(B_i^{(n)} > A_j^{(n)}) = \frac{1}{2},$$

so that

$$\begin{aligned}
\mathbb{E}\left[\mathcal{A}_n\right] &= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\chi_{A_i^{(n)} > B_j^{(n)}}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\chi_{A_i^{(n)} > B_j^{(n)}}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{P}(A_i^{(n)} > B_j^{(n)}) \\
&= \frac{n^2}{2}
\end{aligned}$$

To compute the second moment of $\mathcal{A}_n$, note that, using the symmetry of the distribution,

$$\mathbb{P}(A_i^{(n)}, A_j^{(n)} > B_k^{(n)}) = \mathbb{P}(A_i^{(n)} > B_j^{(n)}, B_k^{(n)}) = \frac{1}{3}$$

$$\begin{aligned}
\mathbb{E}\left[\mathcal{A}_n^2\right] &= \mathbb{E}\left[\left(\sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\chi_{A_{i_1}^{(n)} > B_{j_1}^{(n)}}\right)\left(\sum_{i_2=1}^{n}\sum_{j_2=1}^{n}\chi_{A_{i_2}^{(n)} > B_{j_2}^{(n)}}\right)\right] \\
&= \sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j_2=1}^{n}\mathbb{E}\left[\left(\chi_{A_{i_1}^{(n)} > B_{j_1}^{(n)}}\right)\left(\chi_{A_{i_2}^{(n)} > B_{j_2}^{(n)}}\right)\right] \\
&= \sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{n}\mathbb{E}\left[\left(\chi_{A_{i_1}^{(n)} > B_{j_1}^{(n)}}\right)\left(\chi_{A_{i_2}^{(n)} > B_{j_2}^{(n)}}\right)\right] \\
&\quad + \sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\mathbb{E}\left[\left(\chi_{A_{i_1}^{(n)} > B_{j_1}^{(n)}}\right)\left(\chi_{A_{i_2}^{(n)} > B_{j_1}^{(n)}}\right)\right] \\
&\quad + \sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{n}\mathbb{E}\left[\left(\chi_{A_{i_1}^{(n)} > B_{j_1}^{(n)}}\right)\left(\chi_{A_{i_1}^{(n)} > B_{j_2}^{(n)}}\right)\right] \\
&\quad + \sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\mathbb{E}\left[\left(\chi_{A_{i_1}^{(n)} > B_{j_1}^{(n)}}\right)\left(\chi_{A_{i_1}^{(n)} > B_{j_1}^{(n)}}\right)\right] \\
&= \sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{n}\mathbb{P}(A_{i_1}^{(n)} > B_{j_1}^{(n)})\mathbb{P}(A_{i_2}^{(n)} > B_{j_2}^{(n)}) \\
&\quad + \sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{n}\mathbb{P}(A_{i_1}^{(n)}, A_{i_2}^{(n)} > B_{j_1}^{(n)})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{n}\mathbb{P}(A_{i_1}^{(n)} > B_{j_1}^{(n)}, B_{j_2}^{(n)})\\
&+\sum_{i_1=1}^{n}\sum_{j_1=1}^{n}\mathbb{P}(A_{i_1}^{(n)} > B_{j_1}^{(n)})\\
&= n^2(n-1)^2\left(\frac{1}{2}\right)^2 + n^2(n-1)\left(\frac{1}{3}\right) + n^2(n-1)\left(\frac{1}{3}\right) + n^2\left(\frac{1}{2}\right)\\
&= (3n^2(n-1)^2 + 8n^2(n-1) + 6n^2)\frac{1}{12}\\
&= (3n^4 - 6n^3 + 3n^2 + 8n^3 - 8n^2 + 6n^2)\frac{1}{12}\\
&= (3n^4 + 2n^3 + n^2)\frac{1}{12},
\end{aligned}
$$

then,

$$
\begin{aligned}
\operatorname{Var}(\mathcal{A}_n) &= \mathbb{E}\left[\mathcal{A}_n^2\right] - \mathbb{E}\left[\mathcal{A}_n\right]^2\\
&= (3n^4 + 2n^3 + n^2)\frac{1}{12} - \left(\frac{n^2}{2}\right)^2\\
&= \frac{n^2(2n+1)}{12}.
\end{aligned}
$$

Analogously,

$$\mathbb{E}\left[\mathcal{A}_n\right] = \mathbb{E}\left[\mathcal{B}_n\right] = \mathbb{E}\left[\mathcal{C}_n\right] = \frac{n^2}{2}$$

and

$$\operatorname{Var}(\mathcal{A}_n) = \operatorname{Var}(\mathcal{B}_n) = \operatorname{Var}(\mathcal{C}_n) = \frac{n^2(2n+1)}{12} =: \sigma^2.$$

Therefore, the variables $\widetilde{\mathcal{A}}_n, \widetilde{\mathcal{B}}_n$ and $\widetilde{\mathcal{C}}_n$ in Equation 1 are just the standardized versions of $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$.

Denote the standard normal distribution by $\mathcal{N}(0,1)$ and let $X, Y, Z \sim \mathcal{N}(0,1)$ be random variable such that $\operatorname{Cov}(X,Y) = \operatorname{Cov}(X,Z) = \operatorname{Cov}(Y,Z) = -1/2$. There is a close relation between the number of victories of the dice and these variables.

Lemma 5.0.1. *The PDF of the multivariate normal (X,Y,Z) is*

$$f(x,y,z) = \frac{1}{3\pi}\mathrm{e}^{-\frac{-2}{9}(x^2+y^2+z^2-xy-xz-yz)}$$

and $(X,Y,Z) \in G$ with probability 1, where $G = \{(x,y,z) : x+y+z=0\}$.

In particular, $\operatorname{Dom}(f) = G$.

Proof. This lemma is a direct consequence of the results in [4]. The covariance matrix of $U = (X, Y, Z)$ is

$$\Sigma = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

Note that the eigenvalues of Σ are $\lambda_1 = \lambda_2 = 3/2$ and $\lambda_3 = 0$. Furthermore, it is easy to verify that $\Sigma^+ = \frac{4}{9}\Sigma$ is the pseudoinverse of Σ. As $\operatorname{rank}\Sigma = 2$ and the mean of U, μ, is the null vector, it follows that the PDF of U is given by

$$\begin{aligned} f(x, y, z) &= \frac{1}{\sqrt{(2\pi)^2\lambda_1\lambda_2}} \mathrm{e}^{-\frac{1}{2}\begin{pmatrix} x & y & z \end{pmatrix}\Sigma^+\begin{pmatrix} x & y & z \end{pmatrix}^T} \\ &= \frac{1}{3\pi}\mathrm{e}^{-\frac{2}{9}(x^2+y^2+z^2-xy-xz-yz)}. \end{aligned}$$

Finally, if $N = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$, then $N^T\Sigma = 0$, so that $N^T U = 0$ with probability 1. □

Corolary 5.0.2. *If α, β and γ are real numbers, the the mth moment of $\alpha X + \beta Y + \gamma Z$ is*

$$\mathbb{E}\left[(\alpha X + \beta Y + \gamma Z)^m\right] = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ (m-1)!!(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma)^{m/2} & \text{if } m \text{ is even.} \end{cases}$$

Proof. This follows from a direct computation of said moments using Isserlis' theorem or even by direct means. □

The following theorem is the main result of this report, and it can be generalized to a much wider range of distributions under certain hypotheses, which will be done in a later paper.

Theorem 5.0.3. *The random vector $(\widetilde{\mathcal{A}}_n, \widetilde{\mathcal{B}}_n, \widetilde{\mathcal{C}}_n)$ converges in distribution to (X, Y, Z) as $n \to \infty$.*

Proof. From Cramér-Wold theorem, it suffices to show that $\alpha\widetilde{\mathcal{A}}_n + \beta\widetilde{\mathcal{B}}_n + \gamma\widetilde{\mathcal{C}}_n \xrightarrow{d} \alpha X + \beta Y + \gamma Z$, for each $\alpha, \beta, \gamma \in \mathbb{R}$. The method of moments is used to prove this, as the normal is a variable uniquely determined by its moments. Let $\bar{\mathcal{A}}_n, \bar{\mathcal{B}}_n$ and $\bar{\mathcal{C}}_n$ be the centered variables, so that $\bar{\mathcal{A}}_n = \sigma\widetilde{\mathcal{A}}_n$ and so on. Thus, if $m \geq 0$ is a whole number,

$$\mathbb{E}\left[(\alpha\widetilde{\mathcal{A}}_n + \beta\widetilde{\mathcal{B}}_n + \gamma\widetilde{\mathcal{C}}_n)^m\right] = \frac{1}{\sigma^m}\mathbb{E}\left[(\alpha\bar{\mathcal{A}}_n + \beta\bar{\mathcal{B}}_n + \gamma\bar{\mathcal{C}}_n)^m\right].$$

Note that $\sigma^m \sim n^{3m/2} \cdot 6^{-m}$, so that it suffices to find a good estimate of

$$\mathbb{E}\left[(\alpha\bar{\mathcal{A}}_n + \beta\bar{\mathcal{B}}_n + \gamma\bar{\mathcal{C}}_n)^m\right]. \tag{2}$$

Define $\mathfrak{a}_{ij} = \alpha(\chi_{A_i^{(n)} > B_j^{(n)}} - 1/2)$, $\mathfrak{b}_{ij} = \beta(\chi_{B_i^{(n)} > C_j^{(n)}} - 1/2)$ and $\mathfrak{c}_{ij} = \gamma(\chi_{C_i^{(n)} > A_j^{(n)}} - 1/2)$. It follows that

$$\alpha\bar{\mathcal{A}}_n = \sum_{1\le i,j\le n} \mathfrak{a}_{ij}, \quad \beta\bar{\mathcal{B}}_n = \sum_{1\le i,j\le n} \mathfrak{b}_{ij}, \quad \gamma\bar{\mathcal{C}}_n = \sum_{1\le i,j\le n} \mathfrak{c}_{ij}.$$

If, furthermore, $V = \{\mathfrak{a}_{11}, \mathfrak{a}_{12}, \dots, \mathfrak{a}_{nn}, \mathfrak{b}_{11}, \dots, \mathfrak{b}_{nn}, \mathfrak{c}_{11}, \dots, \mathfrak{c}_{nn}\}$, then

$$\mathbb{E}\left[(\alpha\bar{\mathcal{A}}_n + \beta\bar{\mathcal{B}}_n + \gamma\bar{\mathcal{C}}_n)^m\right] = \sum_{e_1\in V} \cdots \sum_{e_m \in V} \mathbb{E}\left[e_1 e_2 \dots e_m\right].$$

The product $e_1 \dots e_m$ can be viewed as a graph $\mathcal{G}$ with vertices in

$$\{A_1^{(n)}, \dots A_n^{(n)}, B_1^{(n)}, \dots B_n^{(n)}, C_1^{(n)}, \dots C_n^{(n)}\}$$

and each e_k is an edge:

- If $e_k = \mathfrak{a}_{ij}$, for some i and j, then e_k links the vertices $A_i^{(n)}$ and $B_j^{(n)}$;
- If $e_k = \mathfrak{b}_{ij}$, for some i and j, then e_k links the vertices $B_i^{(n)}$ and $C_j^{(n)}$;
- If $e_k = \mathfrak{c}_{ij}$, for some i and j, then e_k links the vertices $C_i^{(n)}$ and $A_j^{(n)}$.

In this case, the graph $\mathcal{G}$ constructed in this way is said to be a σ-graph. It is an important fact that no two vertices of the "same type" are linked; that is, there is no edge of the type $\{A_i^{(n)}, A_j^{(n)}\}$ and so on. In other words, the graph $\mathcal{G}$ is tripartite by the sets $A^{(n)}, B^{(n)}$ and $C^{(n)}$. It is also important to note that some edges can appear multiple times in $\mathcal{G}$. Finally, if M is a constant such that $|\alpha^p\beta^q\gamma^r| < M$ for every $0 \le p, q, r \le m$, which does not depend on n, from the fact that $|\mathfrak{a}_{ij}/\alpha|, |\mathfrak{b}_{ij}/\beta|, |\mathfrak{c}_{ij}/\gamma| \le 1$ for each i, j, it follows that $|e_1 \dots e_m| \le M$, so that

$$|\mathbb{E}\left[e_1 \dots e_m\right]| \le M,$$

for any choice of edges e_i.

Suppose that $\mathcal{G}$ has t connected components, where $t \le m$, possibly with repeated edges, and, as the product of e_i commutes, let $e_1 \dots e_m = p_1 p_2 \dots p_t$, where each p_k is a different connected component of $\mathcal{G}$. Then

$$\mathbb{E}\left[e_1 e_2 \dots e_m\right] = \mathbb{E}\left[p_1\right]\mathbb{E}\left[p_2\right]\dots\mathbb{E}\left[p_t\right].$$

In fact, if $i \ne j$, then p_i and p_j have no vertices in common, and thus they are independent variables in the probability sense.

Now, if e_i is an isolated edge, that is, e_i forms a connected component on its own (with no multiplicity), as $\mathbb{E}\left[e_i\right] = 0$, then $\mathbb{E}\left[e_1 e_2 \dots e_m\right] = 0$.

Claim 1. *There are at most $K_1 n^{(3m-1)/2}$ σ-graphs with less than $m/2$ connected components, where K_1 does not depend on n.*

Let us find an upper bound on the number of σ-graphs with t connected components. First, choose the first edge of each component. For the first component, the first edge can be chosen in at most $9n^2$ ways (a maximum of $3n$ ways for the first vertex of the edge and a maximum of $3n$ ways for the second vertex). Similarly, there are at most $9n^2$ options for the first edge of the second component, and so on. In total, there are no more than $(9n^2)^t = (3n)^{2t}$ ways to choose the first edge of each component. Now the remaining $m - t$ edges need to be distributed. The first of these must be connected to some edge that has already been chosen (the graph cannot have any more components), so the first vertex has up to $2t$ possibilities, and the second has $3n$ possibilities, for a total of $3n(2t)$ choices. For the second edge of the remaining ones, there are no more than $2t+1$ possibilities for the first vertex (the edge allocated previously can add at most one possibility), and no more than $3n$ options for the second vertex, for a total of $3n(2t+1)$ options. Repeating this argument recursively, for the last of the $m - t$ edges, there are a maximum of $3n(2t + (m - t - 1))$ scenarios. In total, there are at most

$$(3n)^{2t} \cdot (3n(2t)) \cdot \cdots \cdot (3n(2t + m - t - 1)) \leq (2m)^m (3n)^{2t+m-t} = (18m)^m n^{m+t}$$

possibilities.

Having chosen the m edges, say, $e_{i_1} \dots e_{i_m}$, they can come from any of the m factors of the power in Equation 2, and thus such a graph can appear up to $m!$ times.

In total, the *sigma*-graphs with t connected components appears no more than $k_1 n^{m+t}$ times, where $k_1 = m! \cdot (18m)^m$. Therefore, there are at most

$$k_1(n^{m+1} + n^{m+2} + \cdots + n^{m+(m-1)/2}) \leq m k_1 n^{(3m-1)/2}$$

σ-graphs with less than $m/2$ connected components (as $t < m/2$, then $t \leq (m - 1)/2$, because t and m are whole numbers). If $K_1 = mk_1$, then K_1 does not depend on n, and the claim is proved.

Claim 2. *If the σ-graph that represents $e_1 \dots e_m$ has more than $m/2$ connected components, then $\mathbb{E}[e_1 \dots e_m] = 0$.*

As there are more than $m/2$ connected components and only m edges, at least one of the components must be an isolated edge, so the claim follows.

If m is odd, each of the σ-graphs must have less than $m/2$ connected components or more than $m/2$ connected components. Therefore

$$\left| \sum_{e_1,\dots,e_m \in V} \mathbb{E}[e_1 \dots e_m] \right| \leq \sum_{t=1}^{\frac{m-1}{2}} \sum_{\substack{e_1,\dots,e_m \in V \\ t}} |\mathbb{E}[e_1 \dots e_m]| + \sum_{t=\frac{m+1}{2}}^{m} \sum_{\substack{e_1,\dots,e_m \in V \\ t}} |\mathbb{E}[e_1 \dots e_m]|$$

$$
\begin{aligned}
&\leq \sum_{t=1}^{\frac{m-1}{2}} \sum_{\substack{e_1,\ldots,e_m \in V \\ t}} M \\
&\leq MK_1 n^{\frac{3m-1}{2}},
\end{aligned}
$$

The index t indicates that the summation extends over the graphs with t connected components. But that implies, by the squeeze theorem, that

$$
\mathbb{E}\left[(\alpha\widetilde{\mathcal{A}}_n + \beta\widetilde{\mathcal{B}}_n + \gamma\widetilde{\mathcal{C}}_n)^m\right] = \frac{1}{\sigma^m} \sum_{e_1,\ldots,e_m \in V} \mathbb{E}\left[e_1 \ldots e_m\right] \to 0.
$$

Suppose, now, that $m = 2k$ for some integer k. A similar argument shows that

$$
\sum_{t=1}^{k-1} \sum_{\substack{e_1,\ldots,e_m \in V \\ t}} \mathbb{E}\left[e_1 \ldots e_m\right] = o(n^{3k}).
$$

There is still a need to consider the case $t = k$. Thankfully, there are more simplifications to be made.

Define the size of a connected component as the number of edges in said component, counting multiplicity.

Claim 3. *If the σ-graph that represents $e_1 \ldots e_{2k}$ has k connected components and some component has size at least 3, then $\mathbb{E}\left[e_1 \ldots e_{2k}\right]$=0.*

If every component has at least two edges, then the total number of edges of $\mathcal{G}$ is a minimum of $3 + 2(k-1) = 2k+1$, a contradiction. Then, at least one of the components has only one edge; that is, at least one edge is isolated.

Claim 4. *There are at most $K_2 n^{3k-1}$ σ-graphs with k connected components of size two and at least one edge with multiplicity two, where K_2 does not depend on n.*

Let d be the number of edges with a multiplicity of two. Once again, there are up to $(3n)^{2k}$ ways to choose the first edges of the k connected components. But now, d components are entirely determined, with $\binom{k}{d} \leq k!$ configurations. For each of the remaining $k-d$ edges, there are up to $6(k-d)n \leq 6kn$ ways to choose from. Thus, including the $(2k)!$ options to choose from the $2k$ factors of the original power, there are no more than $k_2 n^{3k-d}$ possibilities, where $k_2 = 9^k \cdot (2k)!k! \cdot (6k)^k$. Consequently, there are at most

$$
k_2(n^{3k-1} + n^{3k-2} + \cdots + n^{2k}) \leq K_2 n^{3k-1}
$$

of such σ-graphs, where $K_2 = kk_2$ does not depend on n.

A connected component is a cherry if composed of two distinct edges. A graph is a cherry graph if all its connected components are cherries. The vertex of degree 2 in a cherry graph will be called a joint, and the other two will be called tips. It follows that a cherry graph has k connected components of size two with no repeating edges.

All the claims together imply that expected values of all the σ-graphs with less than k connected components or with k connected components, each of size two and with at least one double edge, amount to, at most, $o(n^{3k})$ and that the σ-graphs with more than k components or with k components, with at least one component having more than two edges, disappear in the summation, that is,

$$\sum_{e_1,\dots,e_{2k}\in V} \mathbb{E}\left[e_1 \dots e_{2k}\right] = \sum_{\substack{e_1,\dots,e_{2k}\in V \\ \text{cherry}}} \mathbb{E}\left[e_1 \dots e_{2k}\right] + o(n^{3k}),$$

where the summation extends over all cherry σ-graphs.

What remains is to count all the cherry graphs and compute their expected values.

A cherry is of the type $\overline{XYZ}$ if its joint is in $Y^{(n)}$ and its tips are in $X^{(n)}$ and $Z^{(n)}$, where each of X, Y, Z are one of the letters A, B, C. There are nine unique types of cherries:

1-A) Type $\overline{CAB}$. It represents the product $\mathfrak{c}_{ki}\mathfrak{a}_{ij}$ (the indices i, j and k do not matter, as all $A_i^{(n)}, B_j^{(n)}$ and $C_k^{(n)}$ are iid), so

$$\begin{aligned}
\mathbb{E}\left[\overline{CAB}\right] &= \mathbb{E}\left[\mathfrak{c}_{ki}\mathfrak{a}_{ij}\right] \\
&= \gamma\alpha\mathbb{E}\left[\left(\chi_{C_k^{(n)}>A_i^{(n)}} - \frac{1}{2}\right)\left(\chi_{A_i^{(n)}>B_j^{(n)}} - \frac{1}{2}\right)\right] \\
&= \gamma\alpha\left(\mathbb{P}(C_k > A_i > B_j) - \frac{1}{4}\right) \\
&= -\frac{1}{12}\gamma\alpha;
\end{aligned}$$

2-A) Type $\overline{BAB}$. It represents the product $\mathfrak{a}_{ij}\mathfrak{a}_{ik}$, where $k \neq j$.

$$\begin{aligned}
\mathbb{E}\left[\overline{BAB}\right] &= \mathbb{E}\left[\mathfrak{a}_{ij}\mathfrak{a}_{ik}\right] \\
&= \alpha^2\mathbb{E}\left[\left(\chi_{A_i^{(n)}>B_j^{(n)}} - \frac{1}{2}\right)\left(\chi_{A_i^{(n)}>B_k^{(n)}} - \frac{1}{2}\right)\right] \\
&= \alpha^2\left(\mathbb{P}(A_i > B_j, B_k) - \frac{1}{4}\right) \\
&= \frac{1}{12}\alpha^2
\end{aligned}$$

3-A) Type $\overline{CAC}$. It represents the product $\mathfrak{c}_{ji}\mathfrak{c}_{ki}$, where $k \neq j$.

$$\begin{aligned}
\mathbb{E}\left[\overline{CAC}\right] &= \mathbb{E}\left[\mathfrak{c}_{ji}\mathfrak{c}_{ki}\right] \\
&= \gamma^2\mathbb{E}\left[\left(\chi_{C_j^{(n)}>A_i^{(n)}} - \frac{1}{2}\right)\left(\chi_{C_k^{(n)}>B_i^{(n)}} - \frac{1}{2}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \gamma^2 \left(\mathbb{P}(C_j, C_k > A_i) - \frac{1}{4} \right) \\
&= \frac{1}{12} \gamma^2
\end{aligned}
$$

1-B) Type $\overline{ABC}$. It represents the product $\mathfrak{a}_{ki}\mathfrak{b}_{ij}$. It is similar to type $\overline{CAB}$.

$$
\begin{aligned}
\mathbb{E}\left[\overline{ABC}\right] &= \mathbb{E}\left[\mathfrak{a}_{ki}\mathfrak{b}_{ij}\right] \\
&= -\frac{1}{12}\alpha\beta;
\end{aligned}
$$

2-B) Type $\overline{CBC}$. It represents the product $\mathfrak{b}_{ij}\mathfrak{b}_{ik}$, where $k \neq j$. It is similar to type $\overline{BAB}$.

$$
\begin{aligned}
\mathbb{E}\left[\overline{CBC}\right] &= \mathbb{E}\left[\mathfrak{b}_{ij}\mathfrak{b}_{ik}\right] \\
&= \frac{1}{12}\beta^2
\end{aligned}
$$

3-B) Type $\overline{ABA}$. It represents the product $\mathfrak{a}_{ji}\mathfrak{a}_{ki}$, where $k \neq j$. It is similar to type $\overline{CAC}$.

$$
\begin{aligned}
\mathbb{E}\left[\overline{ABA}\right] &= \mathbb{E}\left[\mathfrak{a}_{ji}\mathfrak{a}_{ki}\right] \\
&= \frac{1}{12}\alpha^2
\end{aligned}
$$

1-C) Type $\overline{BCA}$. It represents the product $\mathfrak{b}_{ki}\mathfrak{c}_{ij}$.

$$
\begin{aligned}
\mathbb{E}\left[\overline{BCA}\right] &= \mathbb{E}\left[\mathfrak{b}_{ki}\mathfrak{c}_{ij}\right] \\
&= -\frac{1}{12}\beta\gamma;
\end{aligned}
$$

2-C) Type $\overline{ACA}$. It represents the product $\mathfrak{c}_{ij}\mathfrak{c}_{ik}$, where $k \neq j$.

$$
\begin{aligned}
\mathbb{E}\left[\overline{ACA}\right] &= \mathbb{E}\left[\mathfrak{c}_{ij}\mathfrak{c}_{ik}\right] \\
&= \frac{1}{12}\gamma^2
\end{aligned}
$$

3-C) Type $\overline{BCB}$. It represents the product $\mathfrak{b}_{ji}\mathfrak{b}_{ki}$, where $k \neq j$.

$$
\begin{aligned}
\mathbb{E}\left[\overline{BCB}\right] &= \mathbb{E}\left[\mathfrak{b}_{ji}\mathfrak{b}_{ki}\right] \\
&= \frac{1}{12}\beta^2
\end{aligned}
$$

Suppose that $n \gg k$ ($n > 2k$ is enough), so there is enough freedom to count the cherry graphs. Given the non-negative integers a_i, b_i, c_i, $i = 1, 2, 3$, fulfilling $\sum_{i=1}^{3} a_i + b_i + c_i = k$, suppose that there are $C(a_1, \ldots, c_3)$ σ-graphs with a_i cherries as described in i-**A**, b_i cherries as described in i-**B** and c_i cherries as described in i-**C**. The expected value of such a graph is

$$\begin{aligned}E(a_1,\dots,c_3) := {}& \mathbb{E}^{a_1}[\overline{CAB}]\mathbb{E}^{a_2}[\overline{BAB}]\mathbb{E}^{a_3}[\overline{CAC}] \\ & \times \mathbb{E}^{b_1}[\overline{ABC}]\mathbb{E}^{b_2}[\overline{CBC}]\mathbb{E}^{b_3}[\overline{ABA}] \\ & \times \mathbb{E}^{c_1}[\overline{BCA}]\mathbb{E}^{c_2}[\overline{ACA}]\mathbb{E}^{c_3}[\overline{BCB}].\end{aligned}$$

Furthermore, given such a graph, all its edges are pairwise distinct so that it appears precisely $(2k)!$ times in the summation. Thus,

$$\sum_{\substack{e_1,\dots,e_{2k}\in V\\ \text{cherry}}} \mathbb{E}\left[e_1\dots e_{2k}\right] = \sum_{a_1+\cdots+c_3=k}(2k)!C(a_1,\dots,c_3)E(a_1,\dots,c_3).$$

Let us calculate $C(a_1,\dots,c_3)$. Choose a_1, a_2 and a_3 distinct vertices from $A^{(n)}$ to be the joints of the cherries of type **1-A**, **2-A** and **3-A**, respectively, and b_1, $2b_3$, c_1 and $2c_2$ to be tips of the cherries of type **1, 3-B** and **1, 2-C**, respectively. There are

$$\binom{n}{a_1,a_2,a_3,b_1,2b_3,c_1,2c_2}$$

possible ways to choose such vertices, where

$$\binom{p}{q_1,\dots,q_i} = \frac{p!}{q_1!\dots q_i!(p-q_1-\cdots-q_i)!}$$

is the multinomial coefficient.

Similarly, there are

$$\binom{n}{b_1,b_2,b_3,c_1,2c_3,a_1,2a_2}$$

and

$$\binom{n}{c_1,c_2,c_3,a_1,2a_3,b_1,2b_2}$$

possible ways to choose the vertices from $B^{(n)}$ and $C^{(n)}$, respectively.

Having chosen the vertices, distribute them to their respective cherries. Fix the vertices chosen to be joints in the natural order $(A_1^{(n)},A_2^{(n)},\dots A_n^{(n)}$ and similarly to $B^{(n)}$ and $C^{(n)}$). Distribute the tips from $A^{(n)}$, and the other cases follow similarly. The tips from $B^{(n)}$ and

$C^{(n)}$ are distributed in a complete analogous fashion. The b_1 vertices chosen to be tips of the type **1-B** have b_1 ways to be distributed to the b_1 chosen from $B^{(n)}$ to be joints. The $2b_3$ vertices chosen to be tips of the type **3-B** have $(2b_3)!/2^{b_3}$ possibilities to be distributed (it is easier to think in terms of multinomial: choose two from the $2b_3$ options to the first of the $2b_3$ vertices from $B^{(n)}$, then two to the next vertex and so on, with

$$\binom{2b_3}{\underbrace{2,\dots,2}_{b_3}} = \frac{(2b_3)!}{2^{b_3}}$$

possibilities). Similarly, there are $c_1!$ (resp. $(2c_2)!/2^{c_2}$) ways to distribute the tips from $A^{(n)}$ of type **1-C** (resp. **2-C**). After distributing all the tips,

$$\begin{aligned}
C(a_1,\dots c_3) &= \binom{n}{a_1,a_2,a_3,b_1,2b_3,c_1,2c_2}\cdot b_1!\cdot\frac{(2b_3)!}{2^{b_3}}\cdot c_1!\cdot\frac{(2c_2)!}{2^{c_2}}\\
&\quad\times\binom{n}{b_1,b_2,b_3,c_1,2c_3,a_1,2a_2}\cdot c_1!\cdot\frac{(2c_3)!}{2^{c_3}}\cdot a_1!\cdot\frac{(2a_2)!}{2^{a_2}}\\
&\quad\times\binom{n}{c_1,c_2,c_3,a_1,2a_3,b_1,2b_2}\cdot a_1!\cdot\frac{(2a_3)!}{2^{a_3}}\cdot b_1!\cdot\frac{(2b_2)!}{2^{b_2}}\\
&= \frac{1}{2^{a_2+a_3+b_2+b_3+c_2+c_3}\prod_{i=1}^{3}[a_i!b_i!c_i!]}\\
&\quad\times\frac{n!}{(n-a_1-a_2-a_3-b_1-2b_3-c_1-2c_2)!}\\
&\quad\times\frac{n!}{(n-b_1-b_2-b_3-c_1-2c_3-a_1-2a_2)!}\\
&\quad\times\frac{n!}{(n-c_1-c_2-c_3-a_1-2a_3-b_1-2b_2)!}.
\end{aligned}$$

Now, see that

$$\begin{aligned}
\frac{n!}{(n-a_1-a_2-a_3-b_1-2b_3-c_1-2c_2)!} &= n^{a_1+a_2+a_3+b_1+2b_3+c_1+2c_2}(1+o(1))\\
&= n^{k+b_3+c_2-b_2-c_3}(1+o(1)),
\end{aligned}$$

in such a manner that

$$\begin{aligned}
C(a_1,\dots,c_3) &= \frac{1}{2^{a_2+a_3+b_2+b_3+c_2+c_3}\prod_{i=1}^{3}[a_i!b_i!c_i!]}\\
&\quad\times n^{k+b_3+c_2-b_2-c_3}\cdot n^{k+c_3+a_2-c_2-a_3}\cdot n^{k+a_3+b_2-a_2-b_3}(1+o(1))\\
&= \frac{n^{3k}(1+o(1))}{2^{k-a_1-b_1-c_1}\prod_{i=1}^{3}[a_i!b_i!c_i!]}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\sum_{\substack{e_1,\dots,e_{2k}\in V\\ \text{cherry}}} \mathbb{E}\left[e_1\dots e_{2k}\right] &= \frac{(2k)!n^{3k}(1+o(1))}{2^k k!} \sum_{a_1+\cdots+c_3=k} \Bigg(\frac{k!}{\prod_{i=1}^{3}[a_i!b_i!c_i!]} \\
&\quad \times (2\mathbb{E}[\overline{CAB}])^{a_1}\mathbb{E}^{a_2}[\overline{BAB}]\mathbb{E}^{a_3}[\overline{CAC}] \\
&\quad \times (2\mathbb{E}[\overline{ABC}])^{b_1}\mathbb{E}^{b_2}[\overline{CBC}]\mathbb{E}^{b_3}[\overline{ABA}] \\
&\quad \times (2\mathbb{E}[\overline{BCA}])^{c_1}\mathbb{E}^{c_2}[\overline{ACA}]\mathbb{E}^{c_3}[\overline{BCB}] \Bigg) \\
&= (2k-1)!!n^{3k}(1+o(1)) \\
&\quad \times \big(2\mathbb{E}[\overline{CAB}] + \mathbb{E}[\overline{BAB}] + \mathbb{E}[\overline{CAC}] \\
&\qquad + 2\mathbb{E}[\overline{ABC}]) + \mathbb{E}[\overline{CBC}] + \mathbb{E}[\overline{ABA}] \\
&\qquad + 2\mathbb{E}[\overline{BCA}] + \mathbb{E}[\overline{ACA}] + \mathbb{E}[\overline{BCB}]\big)^k \\
&= (2k-1)!!n^{3k}(1+o(1)) \\
&\quad \times \Bigg(-\frac{\gamma\alpha}{6} + \frac{\alpha^2}{12} + \frac{\gamma}{12} \\
&\qquad -\frac{\alpha\beta}{6} + \frac{\beta^2}{12} + \frac{\alpha^2}{12} \\
&\qquad -\frac{\beta\gamma}{6} + \frac{\gamma^2}{12} + \frac{\beta^2}{12} \Bigg)^k \\
&= \frac{n^{3k}}{6^k}(2k-1)!!(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\alpha\gamma-\beta\gamma)^k(1+o(1)),
\end{aligned}
$$

and convergence of the moment follows. □

By examining the previous arguments, the reader can understand the methodology used. This can then be applied to similar situations with other types of distributions.

A direct application of the previous theorem is that the intransitive strings become quite rare when n grows. In fact, the proportion of intransitive strings goes to 0.

Corolary 5.0.4. $\lim_{n\to\infty} \mathcal{I}(n)/\mathcal{D}(n) = 0$

Proof. From Equation 1 and Theorem 5.0.3, it follows that

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\mathcal{I}(n)}{\mathcal{D}(n)} &= \mathbb{P}(X,Y,Z>0) + \mathbb{P}(X,Y,Z<0) \\
&\leq \mathbb{P}((X,Y,Z)\in G^c) \\
&= 0,
\end{aligned}
$$

as $(x,y,z)\in G^c \iff x+y+z\neq 0$. □

Actually, even the proportion of neutral string tends to 0 by a similar argument.

6 Conclusion

The study of intransitive dice was facilitated by defining good models to represent the mathematical objects at play. In particular, the first results in this article were a consequence of the string model employed. Using this helpful model, the existence of such intransitive dice was fully characterized, that is, it was possible to determine the existence of an intransitive family of dice for any given number of dice and sides for each die.

When defining models, it is necessary to impose certain restrictions on the set of dice. For example, the initial string model employed in this study did not allow for repeated values. To address this limitation, we introduced the weighted dice model, which provides an analytical expression for comparing any two dice. This model allowed us to fully characterize the existence of intransitive families of dice, similarly to what was achieved with the previous model.

Having studied the existence of intransitive families of dice, new questions naturally arose. One such question is the proportion of intransitive families in a given set of dice. Progress was achieved in this problem by analyzing the model of random dice, which simulates a random draw of dice from a set. This problem was then studied using two different approaches, including a reinterpretation of the string model.

The string model allowed us to computationally explore the problem, which was crucial for understanding the proportion of intransitive families of three dice, given by $\mathcal{I}(n)/\mathcal{D}(n)$, which in turn allowed us to find an asymptotic expression for the probability of three dice being intransitive, but still with open problems to be solved.

The second approach focused on studying the number of wins of one die over another, which is a random variable in the random dice model. This approach allowed us to demonstrate a variation of the Central Limit Theorem with important consequences for the string model.

In conclusion, these different models and approaches to solving the problems highlight the importance of being open to using techniques from other areas of mathematics, even when the areas don't seem to have a clear relationship at first. This is especially evident in the demonstration of Theorem 5.0.3, which utilized graphs to simplify the problem of estimating the moments of the random variable $\alpha\widetilde{\mathcal{A}}_n + \beta\widetilde{\mathcal{B}}_n + \gamma\widetilde{\mathcal{C}}_n$.

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